Exit problem of a two-dimensional risk process from a cone: exact and asymptotic results

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October 2, 2006

Abstract

Consider two insurance companies (or two branches of the same company) that divide between them both claims and premia in some specified proportions. We model the occurrence of claims according to a renewal process. One ruin problem considered is that when the corresponding two-dimensional risk process first leaves the positive quadrant; another is that of entering the negative quadrant. When the claims arrive according to a Poisson process we obtain a closed form expression for the ultimate ruin probability. In the general case we analyze the asymptotics of the ruin probability when the initial reserves of both companies tend to infinity, both under Cramér light-tail and under subexponential assumptions on the claim size distribution.

Key words: First time passage problem, Lévy process, fluctuation theory, exponential asymptotics, subexponential distribution.

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1 Introduction and main results

The multidimensional renewal risk model. In collective risk theory the reserves process X of an insurance company is modeled as

$$X(t) = x + pt - S(t), \tag{1}$$

where x denotes the initial reserve, p is the premium rate per unit of time and $S(t)$ is a stochastic process modeling the amount of cumulative claims up to time t. Taking S to be a compound Poisson or compound renewal process yields the Cramér-Lundberg model and the Sparre-Andersen model, respectively.

Recently, several authors have studied extensions of classical risk theory towards a multidimensional reserves model (1) where $X(t), x, p$ and $S(t)$ are vectors, with possible dependence between the components of $S(t)$. Indeed, the assumption of independence of risks may easily fail, for example in the case of reinsurance, when incoming claims have an impact on both insuring companies at the same time. In general, one can also consider situations where each claim event might induce more than one type of claim in an umbrella policy (see Sundt (1999)). For some recent papers considering dependent risks, see Dhaene and Goovaerts (1996, 1997), Goovaerts and Dhaene (1996), Müller (1997a,b), Denuit et al. (1999), Ambagaspitiya (1999), Dhaene and Denuit (1999), Hu and Wu (1999) and Chan et al. (2003).

Model and problem. In this paper we consider a particular two-dimensional risk model in which two companies split the amount they pay out of each claim in proportions δ_1 and δ_2 where $\delta_1 + \delta_2 = 1$, and receive premiums at rates c_1 and c_2, respectively. Let U_i denote the risk process of the i'th company

$$U_i(t) := -\delta_i S(t) + c_i t + u_i, \qquad i = 1, 2\ ,$$

where u_i denotes the initial reserve and

$$S(t) = \sum_{i=1}^{N(t)} \sigma_i \tag{2}$$

for $N(t)$ being a renewal process with i.i.d. inter-arrival times ζ_i, and the claims σ_i are i.i.d. random variables independent of $N(t)$, with the distribution function $F(x)$. We shall denote by λ and μ the reciprocals of the means of ζ_i and σ_i, respectively. We shall assume that the second company, to be called reinsurer, receives less premium per amount paid out, i.e.:

$$p_1 = \frac{c_1}{\delta_1} > \frac{c_2}{\delta_2} = p_2. \tag{3}$$

On the other hand, the reinsurer needs to have larger reserves than the insurer, as in (12).

As usual in risk theory, we assume that $p_i > \rho := \frac{\lambda}{\mu} = E\sigma/E\zeta$, which implies that $U_i(t) \to \infty$ a.s. as $t \to \infty$ $(i = 1, 2)$. Several ruin problems may be of interest. We consider here:

1. The first time τ_{or} when (at least) one insurance company is ruined, i.e. the exit time of $(U_1(t), U_2(t))$ from the positive quadrant:

$$\tau_{\text{or}}(u_1, u_2) := \inf\{t \geq 0 : U_1(t) < 0. \quad \text{or} \quad U_2(t) < 0\}, \tag{4}$$

   The associated ultimate/perpetual ruin probability will be denoted by

$$\psi_{\text{or}}(u_1, u_2) = P_{(u_1,u_2)}\left[\tau_{\text{or}}(u_1, u_2) < \infty\right], \tag{5}$$

   where $P_{(u_1,u_2)}$ denotes the probability measure conditioned on the event that the initial capitals are equal to (u_1, u_2), i.e. $(U_1(0), U_2(0)) = (u_1, u_2)$.

2. The first time τ_{sim} when the insurance companies experience simultaneous ruin, i.e. the entrance time of $(U_1(t), U_2(t))$ into the negative quadrant:

$$\tau_{\text{sim}}(u_1, u_2) := \inf\{t \geq 0 : U_1(t) < 0 \quad \text{and} \quad U_2(t) < 0\}. \tag{6}$$

   The associated ultimate ruin probability will be denoted by

$$\psi_{\text{sim}}(u_1, u_2) = P_{(u_1,u_2)}\left[\tau_{\text{sim}}(u_1, u_2) < \infty\right]. \tag{7}$$

Let $\tau_i(u_i) = \inf\{t \geq 0 : U_i(t) < 0\}, i = 1, 2$. We will consider also

$$\psi_{\text{both}}(u_1, u_2) = P_{(u_1,u_2)}\left[(\tau_1(u_1) < \infty) \quad \cap \quad (\tau_2(u_2) < \infty)\right]. \tag{8}$$

Clearly:

$$\psi_{\text{sim}}(u_1, u_2) \leq \psi_{\text{both}}(u_1, u_2) = \psi_1(u_1) + \psi_2(u_2) - \psi_{\text{or}}(u_1, u_2), \tag{9}$$

where

$$\psi_i(u) := P_u(\tau_i(u) < \infty) \tag{10}$$

denotes the ruin probability of U_i when $U_i(0) = u$. [1]

Geometrical considerations and solution in the lower cone $\mathcal{C}$. The solution of the degenerate two-dimensional ruin problem (5) strongly depends on the relative sizes of the proportions $\boldsymbol{\delta} = (\delta_1, \delta_2)$ and premium rates $\boldsymbol{c} = (c_1, c_2)$. – see Figure 1. If, as assumed throughout, the angle of the vector $\boldsymbol{\delta}$ with the u_1 axis is bigger than that of $\boldsymbol{c}$, i.e. $\delta_2 c_1 > \delta_1 c_2$, we note that starting with initial capital $(u_1, u_2) \in \mathcal{C}$ in the cone $\mathcal{C} = \{(u_1, u_2) : u_2 \leq (\delta_2/\delta_1)u_1\}$ situated below the line $u_2 = (\delta_2/\delta_1)u_1$, the process (U_1, U_2) will be subject to a "sim/or" ruin precisely at the first crossing time $\tau_i(u_i)$ the u_i axis. Thus, in the domain $\mathcal{C}$ "sim/or" ruin occurs iff there is ruin in the one-dimensional problem corresponding to the risk process U_i with premium c_i and claims $\delta_i\, \sigma$, and the solutions in the lower cone $\mathcal{C}$ coincide with the ultimate ruin probabilities $\psi_i(u_i)$ of the classical risk processes $U_i(t)$:

$$\psi_{\text{or}}(u_1, u_2) = \psi_2(u_2), \qquad \psi_{\text{sim}}(u_1, u_2) = \psi_1(u_1). \tag{11}$$

[1] Note that our notation is slightly different from that of Cai and Li (2005): their ψ_{and} became our ψ_{both}.

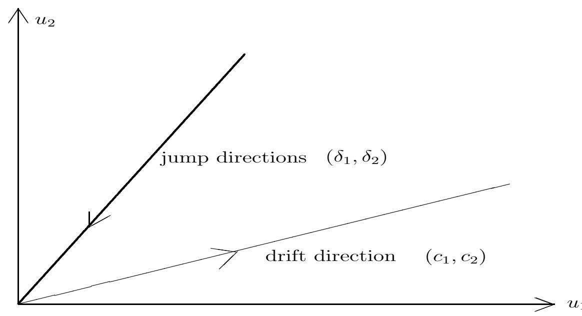

Figure 1: Geometrical considerations

In the opposite case

$$u_2 > (\delta_2/\delta_1)u_1 \tag{12}$$

the solution is more complicated. Note that this is precisely the case of interest for reinsurance.

Solution in the upper cone $\mathcal{C}^c$: piecewise linear barriers A key observation is that the "or" and "sim" ruin times τ in (4), (6) are also equal to

$$\tau(u_1, u_2) = \inf\{t \geq 0 : S(t) > b(t)\}, \tag{13}$$

where

$$b(t) = b_{\min}(t) = \min\{(u_1 + c_1 t)/\delta_1, (u_2 + c_2 t)/\delta_2\}$$

in the "or" case and

$$b(t) = b_{\max}(t) = \max\{(u_1 + c_1 t)/\delta_1, (u_2 + c_2 t)/\delta_2\}$$

in the "sim" case. Our two dimensional problem may thus also be viewed as a one dimensional crossing problem over a piecewise linear barrier. Note the relation to asset-liability management models, in which regulatory requirements impose prescribed limits of variation for the difference between the assets $P(t)$ of a company and its liabilities $S(t)$ (see Gerber and Shiu (2003)), and which also translate typically into (several) linear barriers.

Note that in the case that the initial reserves satisfy $(u_1, u_2) \in \mathcal{C}$, that is, $u_2/\delta_2 \leq u_1/\delta_1$, the barriers $b_{\min}(t) := (u_2 + c_2 t)/\delta_2, b_{\max} := (u_1 + c_1 t)/\delta_1$ are linear and we see again that "or" and "sim" ruin always happen for the second and first company respectively. Thus, in the case $u_1 \leq u_2$, explicit formulas for $\psi_{\text{sim}}, \psi_{\text{or}}$ and ψ_{both} directly follow from the theory of one-dimensional ruin (see e.g. Rolski et al. (1999) or Asmussen (2000)). In the case $u_2 > u_1$, which is authentically two-dimensional, and if N is a Poisson process, we shall obtain (in Section 3) closed form solutions for $\psi_{\text{sim}}, \psi_{\text{or}}$ and ψ_{both} in terms of one-dimensional ruin functions (even though typically, multi-dimensional ruin problems do not admit analytic solutions).

Scaling. Let

$$X_i(t) := U_i(t)/\delta_i$$

denote risk processes with drift p_i, claims σ_k and initial capitals $x_i = u_i/\delta_i, i = 1, 2$, and note that the process $(X_1(t), X_2(t))$ has the same ruin probability as the original two-dimensional process $(U_1(t), U_2(t))$. Thus, by scaling, it suffices to analyze the case when $\delta_1 = \delta_2 = 1$ (and $c_1 = p_1$, $c_2 = p_2$). Our model becomes thus the particular case of (1) in which the components of the claims are equal.

In the future, we will write $\psi_{\text{or/sim/both}}(x_1, x_2)$ for $\psi_{\text{or/sim/both}}(u_1, u_2)$, with $x_i = u_i/\delta_i$, and $S(t)$ for the common value of $S_i(t)$ $(i = 1, 2)$ in the model (1).

Asymptotics. We next turn to the asymptotic behaviour of the different ruin probabilities $\psi_{\text{or/sim/both}}(x_1, x_2)$ as the initial capitals x_1, x_2 tend to infinity along a ray (i.e. x_1/x_2 is constant). In this paper we shall consider two different situations: the "light-tailed" case when the Cramér-assumption holds and the case when the distribution of σ belongs to the class of subexponential distributions.

Cramér/Light-tailed case.

Definition 1 We will say that a two-dimensional renewal risk process $X = (X_1, X_2)$ satisfies the Cramér-conditions if there exist constants $\gamma_i > 0$ $(i = 1, 2)$ such that

$$E[\mathrm{e}^{\gamma_i(\sigma_1 - p_i\zeta_1)}] = 1. \tag{14}$$

Note that while stronger assumptions have been used in the literature, in our essentially one-dimensional setup – see (13) – a local condition at γ_1 will suffice – see Theorem 1.

In the case that S is a compound Poisson process the Cramér condition can equivalently be formulated in terms of the characteristic exponent $\kappa_i(s) = t^{-1}\log E[\mathrm{e}^{sX_i(t)}] = p_i s + \lambda(E[\mathrm{e}^{-\sigma s}] - 1)$ of $X_i(t) = p_i t - S(t)$, as the existence of $\gamma_i > 0$ $(i = 1, 2)$ such that

$$\kappa_i(-\gamma_i) = 0. \tag{15}$$

By (11) the solution in the lower cone $\mathcal{C}$ of the "or/sim" problems coincide with the ultimate ruin probabilities $\psi_i(x_i)$ of the processes $X_i(t)$ $i = 1, 2$. Letting thus the initial reserves x_1, x_2 of both companies tend to infinity within the cone $\mathcal{C}$ along a ray $x_1/x_2 = a$ with $a \geq 1$, we find by the Cramér-Lundberg approximation that

$$\lim_{K\to\infty} \frac{\psi_{\text{sim/both}}(aK, K)}{\mathrm{e}^{-\gamma_1 aK}} = C_1, \qquad \lim_{K\to\infty} \frac{\psi_{\text{or}}(aK, K)}{\mathrm{e}^{-\gamma_2 K}} = C_2, \tag{16}$$

where C_1, C_2 are non-negative finite constants that, in the case that S is a compound Poisson process, are given explicitly by

$$C_i = -\kappa_i'(0)/\kappa_i'(-\gamma_i). \tag{17}$$

Next we turn to the asymptotic behaviour of $\psi_{\text{or/sim/both}}(x_1, x_2)$ if the initial reserves tend to infinity along a ray $x_1/x_2 = a$ with $a < 1$. In this case we find different asymptotic results within sub-cones, as typical in such cases; see for example Borovkov and Mogulskii (2001) and Ignatyuk et al. (1994). We identify two cones $\mathcal{D}_i, i = 1, 2$, adjoining the boundary $x_1 = 0$, throughout which or/sim ruin are due to the first/second company in $\mathcal{D}_1/\mathcal{D}_2$, and to the second/first company in $\overline{\mathcal{D}}_2^c/\overline{\mathcal{D}}_1^c$, respectively. More precisely, these cones

$$\mathcal{D}_1 = \{(x_1, x_2) \in \mathcal{C}^c : x_1 < x_2 s_1\}, \quad \mathcal{D}_2 = \{(x_1, x_2) \in \mathcal{C}^c : x_1 < x_2 s_2\} \tag{18}$$

are separated by the rays $\mathcal{R}_{s_1} = \{x_1 = s_1 x_2\}$, $\mathcal{R}_{s_2} = \{x_1 = s_2 x_2\}$ with

$$s_1 = \frac{\kappa_1'(-\gamma_1)}{\kappa_2'(-\gamma_1)}, \qquad s_2 = \left(\frac{\kappa_1'(-\gamma_2)}{\kappa_2'(-\gamma_2)}\right)_+. \tag{19}$$

In Section 4.2 below it will be verified that $s_2 < s_1 < 1$, so that the rays $\mathcal{R}_{s_1}$ and $\mathcal{R}_{s_2}$ are disjoint subsets of $\mathcal{C}^c$. For a more detailed intuitive explanation of the different behavior in these sub-cones, see the remark following Lemma 4.

Note that our results are sometimes sharper than the classical ones – see Theorem 7, and that they are obtained directly from classical one-dimensional convergence results in renewal theory, obtained by Arfwedson (1955) and Höglund (1990).

Throughout the paper we write $f(x) \sim h(x)$ as $x \to \infty$ if $\lim_{x\to\infty} f(x)/h(x) = 1$, $f(x) \sim^{\log} h(x)$ if $\log f(x) \sim \log h(x)$ and $f(x) = \mathrm{o}(h(x))$ as $x \to \infty$ if $\lim_{x\to\infty} f(x)/h(x) = 0$. Write Θ_i, $i = 1, 2$ for the domain of κ_i, the set of $x \in \mathbb{R}$ for which $\kappa_i(x)$ is finite.

Theorem 1 *Let S be a compound Poisson process as in (2) and let $a < 1$. Assume that (15) holds and let $\kappa_1'(-\gamma_1) > -\infty$.*

(a) It holds that, as $K \to \infty$,

$$\begin{aligned} \psi_{\mathrm{or}}(aK, K) &\sim C_2 \mathrm{e}^{-\gamma_2 K} + C_1 \mathrm{e}^{-\gamma_1 a K}, \\ \psi_{\mathrm{both}}(aK, K) &= \mathrm{o}(C_2 \mathrm{e}^{-\gamma_2 K} + C_1 \mathrm{e}^{-\gamma_1 a K}). \end{aligned}$$

(b) Assume that there exists an $x \in \operatorname{int}\Theta_2$ with $\kappa_2'(x) = \frac{p_1 - p_2}{1-a}$. As $K \to \infty$,

$$\psi_{\mathrm{sim}}(aK, K) \sim \psi_{\mathrm{both}}(aK, K) \sim \begin{cases} C_1 \mathrm{e}^{-\gamma_1 a K}, & s_1 < a < 1 \\ C_2 \mathrm{e}^{-\gamma_2 K}, & 0 < a < s_2, \end{cases}$$

$$\psi_{\mathrm{both}}(aK, K) = \mathrm{o}(\min\{C_1 \mathrm{e}^{-\gamma_1 a K}, C_2 \mathrm{e}^{-\gamma_2 K}\}), \qquad a \in (s_2, s_1),$$

where the region $0 < a < s_2$ is understood to be empty if $s_2 = 0$. Further, if $s_2 > 0$ and $a \in (s_2, s_1)$, $\psi_{\mathrm{sim}}(aK, K) \sim^{\log} \psi_{\mathrm{both}}(aK, K)$.

The asymptotics in part (a) do not depend on the particular structure of the risk process considered here; in Theorem 5 we shall show that these asymptotics remain valid for a general two-dimensional Lévy process. This extension implies that the asymptotics in Theorem 1(a) also remain valid if the cumulative claims process S is taken as in (2) with a general renewal process N. Indeed, note that the ruin probabilities $\psi_{\text{or/both/sim}}$ do not change if we replace $X = (X_1, X_2)$ by a two-dimensional compound Poisson process with unit rate and jump sizes distributed as $(\sigma_n - p_1\zeta_n, \sigma_n - p_2\zeta_n)$.

To show the asymptotics in (b) we shall exploit the special (degenerate) structure of the risk process in (2). In Theorem 7 of Section 4.3, we obtain even sharper asymptotics (a two term asymptotic expansion) in the case that S is a compound Poisson process with exponential jumps.

Subexponential case. For any distribution function G on $[0,\infty)$ we denote the tail by $\overline{G}(x) = 1 - G(x)$. The distribution G is called *subexponential* if $\overline{G}(x) > 0$ for all x and

$$\lim_{x\to\infty} \overline{G^{*2}}(x)/\overline{G}(x) = 2 \ ,$$

where G^{*2} is the fold of G with itself. A distribution function G on $\mathbb{R}$ belongs to the class S^* introduced by Klüppelberg (1988) if and only if $\overline{G}(x) > 0$ for all x and

$$\int_0^x \overline{G}(x-y)\overline{G}(y)\,\mathrm{d}y \sim 2\nu\overline{G}(x), \quad \text{as } x\to\infty, \tag{20}$$

where

$$\nu = \int_0^\infty \overline{G}(x)\,\mathrm{d}x.$$

Note that any distribution from S^* is subexponential. Its integrated tail distribution G_I is defined as $G_I(x) = \nu^{-1}\int_0^x \overline{G}(y)\mathrm{d}y$. The subexponential distributions model claim sizes resulting from catastrophic events like earthquakes, storms, terrorist attacks etc. Insurance companies use the lognormal distribution (which is subexponential) to model car claims – see Rolski et al. (1999) for details. Let F denote the distribution of the claim size σ and define

$$H(aK,K) = \int_0^\infty \overline{F}\left(\max\{aK + m_1 t, K + m_2 t\}\right)\,\mathrm{d}t \tag{21}$$

for $m_i = p_i E[\zeta] - E[\sigma]$ $(i = 1,2)$. If $a \geq 1$ and if F_I is subexponential distribution, then it follows from the one-dimensional reduction and Embrechts and Veraverbeke (1982) (see also Zachary (2004)) that

$$\lim_{K\to\infty} \frac{\psi_{\text{or}}(aK,K)}{\overline{F}_I(K)} = \frac{1}{m_2\mu}, \quad \lim_{K\to\infty} \frac{\psi_{\text{sim}}(aK,K)}{\overline{F}_I(aK)} = \frac{1}{m_1\mu},$$

where $\mu = E[\sigma]^{-1}$. If $a < 1$ the asymptotics of $\psi_{\text{or/sim/both}}$ are as follows:

Theorem 2 *If $a < 1$, $F \in S^*$ and $E[\zeta^{2+\delta}] < \infty$ for some $\delta > 0$, then it holds that*

$$\psi_{\text{or}}(aK,K) \sim \frac{1}{m_1\mu}\overline{F}_I(aK) + \frac{1}{m_2\mu}\overline{F}_I(K) - H(aK,K)\ , \tag{22}$$

$$\psi_{\text{sim}}(aK,K) \sim \psi_{\text{both}}(aK,K) \sim H(aK,K) \tag{23}$$

as $K\to\infty$.

Note that, in contrast to the case of light tails, the asymptotic probability of joint ruin of both companies at the same time appears in the asymptotics for $\psi_{\mathrm{or}}(aK, K)$ (and is equal to $H(aK, K)$). This is what we might expect — "large" claims cause often bankruptcy not only of the insurance company but of a whole chain of reinsurers.

Contents. The rest of the paper is organised as follows. Section 2 is devoted to auxiliary results regarding one-dimensional first passage times for spectrally negative Lévy processes. In Section 3 an explicit form of the ruin probabilities (5) and (7) is derived in the case that N is a Poisson process. The proofs of the asymptotics for the Cramèr case of light-tailed claims, and the subexponential case are presented in Sections 4 and 5, respectively.

2 Preliminaries

Let X be a spectrally negative Lévy process, i.e. a stochastic process with càdlàg paths without positive jumps that has stationary independent increments defined on some probability space $(\Omega, \mathcal{F}, P)$ that satisfies the usual conditions. By $(P_x, x \in \mathbb{R})$ we denote the family of measures conditioned on $\{X(0) = x\}$ with $P_0 = P$. We exclude the case that X has monotone paths. If X is of bounded variation, X takes the form of $X(t) = pt - S(t)$, where $S(t)$ is a subordinator and $p > 0$ is called the infinitesimal drift of X.

By the absence of positive jumps, the moment generating function of $X(t)$ exists for all $\theta \geq 0$ and is given by

$$E\big[\mathrm{e}^{\theta X(t)}\big] = \exp\big(t\,\kappa(\theta)\big), \qquad \theta \geq 0$$

for some function $\kappa(\theta)$, called Laplace exponent, which is well defined at least on the positive half axis, where it is convex with the property $\lim_{\theta\to\infty} \kappa(\theta) = +\infty$. Let $\Phi(0)$ denote its largest root. On $[\Phi(0), \infty)$ the function κ is strictly increasing and we denote its right-inverse function by $\Phi : [0, \infty) \to [\Phi(0), \infty)$.

Each measure $P^{(c)}$ in the exponential family of measures $\mathcal{P} := \{P^{(c)} : c \text{ such that } \kappa(c) < \infty\}$ is defined by its the Radon-Nikodym derivative $\Lambda^{(c)}$

$$\left.\frac{\mathrm{d}P^{(c)}}{\mathrm{d}P}\right|_{\mathcal{F}_t} = \Lambda^{(c)}(t) = \exp\left(cX(t) - \kappa(c)t\right). \tag{24}$$

By $P_x^{(c)}$ we shall denote the translation of $P^{(c)}$ under which $X(0) = x$. The characteristic function of the process X under the measure $P^{(c)} \in \mathcal{P}$ is given by

$$\kappa_c(\theta) = \kappa(c + \theta) - \kappa(c). \tag{25}$$

2.1 Cramér asymptotics

Results concerning spectrally negative Lévy processes are conveniently expressed in terms of the so-called scale function $W^{(q)} : [0, \infty) \to [0, \infty)$ defined by its

Laplace transform:

$$\int_0^\infty e^{-\alpha x} W^{(q)}(x)\, dx = \frac{1}{\kappa(\alpha) - q},$$

where $\kappa(\alpha)$ is the Laplace exponent of $X(t)$. As may be seen from the fluctuation theory of spectrally negative Lévy processes (e.g. Bertoin (1996), Bingham (1975)) the Laplace transform in t of the survival probability up to time t:

$$\overline{\psi}(x,t) = 1 - \psi(x,t) = 1 - P_x(\tau < t),$$

for

$$\tau = \inf\{t \geq 0 : X(t) < 0\}$$

is for $x \geq 0$ equal to

$$\overline{\Psi}^{(q)}(x) = \Phi(q)^{-1} W^{(q)}(x) - \overline{W}^{(q)}(x), \tag{26}$$

where $W^{(q)}$ denotes the corresponding scale function and its anti-derivative $\overline{W}^{(q)}(x) = \int_0^x W^{(q)}(y)\mathrm{d}y$. Letting $q \downarrow 0$ in the expression $q\overline{\Psi}^{(q)}$ we recover that, if $\kappa'(0) > 0$,

$$\psi(x) = \psi(x, \infty) = 1 - W(x)/W(\infty), \tag{27}$$

where, by a Tauberian theorem, $W(\infty) := \lim_{x\to\infty} W(x) = \frac{1}{\kappa'(0^+)} = \Phi'(0^+)$. The Cramér-Lundberg approximation states that, if there exists a $\gamma > 0$ with $\kappa(-\gamma) = 0$ and if X is a classical risk process, it holds that

$$\lim_{x\to\infty} e^{\gamma x}\psi(x) = C = -\kappa'(0)/\kappa'(-\gamma), \tag{28}$$

where C is understood to be zero if $\kappa'_+(-\gamma) = -\infty$, where κ'_+ denotes the right-derivative of κ. Bertoin and Doney (1994) showed that this result remains valid if X is a Lévy process (not just spectrally negative). The parameter γ is called the adjustment coefficient.

The following result, given for later reference, concerns the expected time of ruin.

Lemma 1 *Suppose that $\kappa'(0^+) \in (-\infty, 0]$. Then, as $x \to \infty$,*

$$x/E_x[\tau] \to -\kappa'(0^+).$$

Proof: Assume first that $\kappa'(0) \in (-\infty, 0)$. Applying the optional stopping theorem to the martingale $X_t - t\kappa'(0^+)$ and the bounded stopping time $\tau \wedge T$ shows that

$$E_x[X_{\tau\wedge T}] - x = E_x[\tau \wedge T]\kappa'(0^+).$$

Since $|X_{\tau\wedge T}|$ is dominated by $S_\tau - X_\tau$, which has finite expectation, and τ is finite P_x-a.s., it follows by letting $T \to \infty$ and invoking the dominated convergence theorem that the previous display remains valid with $\tau \wedge T$ replaced by τ. The assertion now follows using that $E_x[X_\tau]$ remains finite as $x \to \infty$.

The case $\kappa(0^+) = 0$ follows by adding a small negative drift to the process. $\square$

Example 1 If, under P, X is a drift p minus a compound Poisson process with rate λ and exponential jump sizes with mean μ, then $\kappa(\theta) = p\theta - \lambda\theta/(\mu + \theta)$ and its scale function $W^{(q)}$ is given by

$$W^{(q)}(x) = p^{-1}\left(A_+ e^{q^+(q)x} - A_- e^{q^-(q)x}\right),$$

where $A_\pm = \frac{\mu + q^\pm(q)}{q^+(q) - q^-(q)}$ with $q^+(q) = \Phi(q)$ and $q^-(q)$ the smallest root of $\kappa(\theta) = q$. Inserting the found expression for $W^{(q)}$ in (26) we see that in this case

$$\overline{\Psi}^{(q)}(x) = q^{-1}[1 - (1 + q^-(q)/\mu)e^{-q^-(q)x}]. \tag{29}$$

From (27) one can verify that, if, as assumed throughout, $p > \frac{\lambda}{\mu}$, then $\psi(x) = Ce^{-\gamma x}$, where the adjustment coefficient is $\gamma = \mu - \lambda/p$ and $C = \lambda/(\mu p)$. Further, under $P^{(c)}$, X is still a drift p minus a compound Poisson process with exponential jumps but with the changed rates $\lambda_c = \lambda\frac{\mu}{\mu+c}$ and $\mu_c = \mu + c$. In particular, $\lambda_{-\gamma} = \mu\, p$ and $\mu_{-\gamma} = \lambda/p$ are the parameters of the corresponding measure.

Example 2 If $X(t) = mt + \sigma B(t)$ where $B(t)$ is standard Brownian motion, then $\kappa(\theta) = \frac{\sigma^2}{2}\theta^2 + m\theta$ and the scale function $W^{(q)}$ is given by

$$W^{(q)}(x) = \left(\frac{2}{\sigma^2(q^+(q) - q^-(q))}\right)\left(e^{q^+(q)x} - e^{q^-(q)x}\right),$$

where $q^+(q) = \Phi(q)$ and $q^-(q)$ is the smallest root of $\kappa(\theta) = q$ and $\overline{\Psi}^{(q)}$ reads as

$$\overline{\Psi}^{(q)}(x) = q^{-1}[1 - e^{-q^-(q)x}]. \tag{30}$$

In particular, if $m < 0$, $\psi(x) = e^{-\gamma x}$, where $\gamma = \frac{2m}{\sigma^2}$ is the adjustment coefficient. Further, under $P^{(c)}$, X is still Brownian motion, but the drift changes to $m + c\sigma^2$. The drift of the measure associated to $c = -\gamma$ is $-m$, i.e. the Brownian motion switches its drift.

2.2 Generalization of ballot theorem

Denoting by

$$I(t) = \inf_{0 \le s \le t} X(s) \wedge 0$$

the infimum of X, the well-known ballot theorem states that, if X has bounded variation, then

$$P\left(X(t) \in dz, I(t) \ge 0\right) = \frac{z}{p\,t} P(X(t) \in dz), \tag{31}$$

where p denotes the infinitesimal drift of X. The next result generalizes the ballot theorem to general starting point $x \ge 0$ and allows for unbounded variation. We shall derive the result under the condition (AC) that the one-dimensional distributions of X are absolutely continuous with respect to the Lebesgue measure

$$P(X(t) \in \mathrm{d}x) \ll \mathrm{d}x, \qquad t > 0. \tag{AC}$$

Proposition 1 *Let X be a spectrally negative Lévy process satisfying (AC) and write $p(t,x)$ for a version of $\frac{P(X(t)\in \mathrm{d}x)}{\mathrm{d}x}$. Then for $t > 0$ and $x, z \geq 0$ it holds that*

$$\begin{aligned} P_x(X(t) \in \mathrm{d}z, I(t) \geq 0) &= P_x(X(t) \in \mathrm{d}z) \\ &- z \int_0^t \frac{1}{(t-s)} P(X(t-s) \in \mathrm{d}z) p(s, -x) \mathrm{d}s. \end{aligned} \tag{32}$$

Let us sketch the idea of the proof of this result. First of all, writing $T(z) = \inf\{t \geq 0 : X(t) > z\}$, note that equation (32) is equivalent to

$$\begin{aligned} P_x(X(t) \in \mathrm{d}z, I(t) < 0) &= \int_0^t p(t-s, -x) P(T(z) \in \mathrm{d}s) \mathrm{d}z \\ &= z \int_0^t \frac{1}{t-s} P(X(t-s) \in \mathrm{d}z) p(s, -x) \mathrm{d}s, \end{aligned} \tag{33}$$

where the second equality follows by Kendall's identity (see (75) below). Heuristically, (33) follows by conditioning on the last time $t - s$ when $X(t)$ crosses 0 before arriving in $\mathrm{d}z$, which is the first time the time-reversed process, starting in z, downcrosses 0, and then using that the time-reversed process, having the same law as $-X$ and thus being spectrally positive, hits 0 when first downcrossing 0.

From this result we can recover the ballot theorem:

Corollary 1 *Let $X(t)$ denote a spectrally negative Lévy process of bounded variation. Then (31) holds true.*

The proofs of Proposition 1 and Corollary 1 are deferred to the Appendix 6.

2.3 Asymptotics of finite time ruin probabilities

For the analysis of the two-dimensional ruin problems we shall need a characterisation of the asymptotics of the finite time ruin probability $\psi(x,t) = P_x[I_t < 0]$ of the spectrally negative Lévy process X as x and t go to infinity according to a ray $x/t = a$. The asymptotics are expressed in terms of the adjustment coefficient γ and the convex conjugate κ^* of κ,

$$\kappa^*(v) = \sup_{\beta\in\mathbb{R}} [v\beta - \kappa(-\beta)]. \tag{34}$$

If $v \in \mathcal{K} := \{-\kappa'(x) : x \in \mathrm{int}\Theta\}$, where $\mathrm{int}\Theta$ is the interior of the domain $\Theta = \{x \in \mathbb{R} : \kappa(x) < \infty\}$ of κ then an explicit expression for $\kappa^*(v)$ read as

$$\kappa_i^*(v) = -\kappa_i'(-\theta_v)\theta_v - \kappa_i(-\theta_v),$$

where $\theta = \theta_v$ is the unique root of $\kappa'(-\theta) = -v$.

Proposition 2 *Suppose that there exists a $\gamma \geq 0$ with $\kappa(-\gamma) = 0$ and $\kappa'(-\gamma) < 0$ and assume that $a \in \mathcal{K}$. Then, as $x \to \infty$,*

$$\begin{aligned} \psi(ax,x) &\sim Ce^{-\gamma ax}, && 0 < a < -\kappa'(-\gamma), \\ \log \psi(ax,x) &\sim -\kappa^*(a)x, && a > -\kappa'(-\gamma), \\ \psi(ax,\infty) - \psi(ax,x) &\sim Ce^{-\gamma ax}, && a > -\kappa'(-\gamma), \\ \log(\psi(ax,\infty) - \psi(ax,x)) &\sim -\kappa^*(a)x, && 0 < a < -\kappa'(-\gamma). \end{aligned}$$

Proof: We only prove the statements regarding $\psi(ax,x)$ as the proof for $\psi(ax,\infty) - \psi(ax,x)$ is similar. Changing measure with $\Lambda^{(-\gamma)}$ and using that $\kappa(-\gamma) = 0$, it follows that

$$\begin{aligned} e^{\gamma ax}\psi(ax,x) &= e^{\gamma ax}P_{ax}(\tau < x) = E^{(-\gamma)}_{ax}[e^{\gamma X(\tau)}\mathbf{1}_{\{\tau<x\}}] \\ &= E^{(-\gamma)}_{ax}[e^{\gamma X(\tau)}] - E^{(-\gamma)}_{ax}[e^{\gamma X(\tau)}\mathbf{1}_{\{\tau>x\}}]. \end{aligned}$$

The first expectation converges to C by (28) and, if $a < -\kappa'(-\gamma)$, the second one converges to zero by the bounded convergence theorem in conjunction with the law of large numbers. Indeed, it holds that

$$\begin{aligned} P^{(-\gamma)}_{ax}(\tau \leq x) &= P^{(-\gamma)}(I(x) \leq -ax) \geq P^{(-\gamma)}(X(x) \leq -ax) \\ &= P^{(-\gamma)}(X(x)/x \leq -a) \end{aligned}$$

and, in view of the strong law of large numbers, the latter probability, converges to 1 as $x \to \infty$.

Next we consider the case $a > -\kappa'(-\gamma)$. Employing $\Lambda^{(-\theta_a)}$ as a change of measure and using that $\theta_a > 0$, $X(\tau) \leq 0$ and $\kappa(-\theta_a) > 0$, we find the upper bound

$$\psi(ax,x) = e^{-a\theta_a x}E^{(-\theta_a)}_{ax}[e^{\theta_a X(\tau)+\kappa(-\theta_a)\tau}\mathbf{1}_{\{\tau<x\}}] \leq e^{-a\theta_a x+\kappa(-\theta_a)x} = e^{-x\kappa^*(a)}.$$

To establish the lower bound we invoke the classical asymptotics of Bahadur and Rao (1960): for $a \in \mathcal{K}$, it holds that $P(X(n) < -an + \epsilon_n) \sim cn^{-1/2}e^{-n\kappa^*(a)}$ for $\epsilon_n = o(\sqrt{n})$, as $n \to \infty$, ($n \in \mathbb{N}$). Therefore, we find that

$$\begin{aligned} \log \psi(ax,x) &= \log P(I(x) < -ax) \geq \log P(X(\lfloor x \rfloor) < -ax) \\ &\sim \log P(X(\lfloor x \rfloor) < -ax) \sim -[x]\kappa^*(a) \sim -x\kappa^*(a), \end{aligned}$$

where $\lfloor x \rfloor$ is the largest integer smaller or equal to x. □

We shall also need the following sharper result that was obtained by Arfwedson (1955). The current statement is from Höglund (1990) (Cor. 2.3).

Theorem 3 *Let $X(t) = ax + pt - S(t)$, where S is a compound Poisson process with positive jumps. Suppose that there exists a $\gamma \geq 0$ with $\kappa(-\gamma) = 0$ and $\kappa'(-\gamma) < 0$ and assume that $a \in \mathcal{K}$. Then, there exists a positive constant D such that, as $x \to \infty$,*

$$\begin{aligned} \psi(ax,x) &\sim Dx^{-1/2}e^{-\kappa^*(a)x}, && a > -\kappa'(-\gamma), \\ \psi(ax,\infty) - \psi(ax,x) &\sim Dx^{-1/2}e^{-\kappa^*(a)x}, && 0 < a < -\kappa'(-\gamma). \end{aligned}$$

If θ_a' is the root of $\kappa(-s)=\kappa(-\theta_a)$, then

$$D=\frac{\theta_a-\theta_a'}{|\theta_a\theta_a'|\sqrt{2\pi\kappa''(-\theta_a)}}. \tag{35}$$

3 Exact two-dimensional ruin probabilities

In this section we propose methods for computing the ruin probabilities of the model with equal claims in the case when $(x_1,x_2)\in\mathcal{C}^c$ and $S(t)$ is a general spectrally positive Lévy process. The generalisation of the ballot theorem derived in the previous section enables us to express the ruin probabilities for general claim size distributions in terms of the one-dimensional distributions in Proposition 4 in Subsection 3.2. Specializing to the case of a compound Poisson process with exponential jumps or a Brownian motion this leads to explicit expressions.

3.1 The probability of crossing a piecewise linear barrier.

In the case $x_2>x_1$, the survival probabilities $\overline{\psi}_{\rm or}(x_1,x_2)/\overline{\psi}_{\rm sim}(x_1,x_2)$ are given by the probability that the process S stays below piecewise linear barriers

$$b_{\min}(t)=\min_{i=1,2}\{x_i+p_it\},\quad b_{\max}(t)=\max_{i=1,2}\{x_i+p_it\}$$

formed by the lines $t\mapsto x_1+p_1t$ and $t\mapsto x_2+p_2t$, which cross at

$$T=T(x_1,x_2)=\frac{x_2-x_1}{p_1-p_2}. \tag{36}$$

In the first case for example, this requires staying below the barrier x_1+p_1t between the times 0 and T and subsequently staying below the barrier x_2+p_2t after time T. In the case when S is Markovian (not necessarily a spectrally positive Lévy process), this yields by conditioning at time T:

$$\overline{\psi}_{or}(x_1,x_2)=\int_0^\infty\overline{\psi}_1(\mathrm{d}z,T|x_1)\overline{\psi}_2(x_2+p_2T-z),$$

where

$$\overline{\psi}_i(\mathrm{d}z,T|x):=P_0(S(t)\le x+p_it,\forall t\in[0,T],S(T)\in dz) \tag{37}$$

is the density at time T of the paths $S(T)$ which "survive" the upper barrier $x+p_it$ and where we used the fact that $x_1+p_1T=x_2+p_2T$. We find it convenient to reformulate this result in terms of the two coordinates of our reserves process

$$X_i(t):=x_i+p_it-S(t),\quad i=1,2,$$

their infima

$$I_i(t)=\inf_{0\le s\le t}X_i(s)\wedge 0$$

and the coordinate-wise densities of the "non-ruined" paths

$$\overline{\psi}_i(\mathrm{d}z, T|x_i) = P_{x_i}(I_i(T) \geq 0, X_i(T) \in dz). \tag{38}$$

We arrive thus at the following result, which relates the survival probability of the two dimensional process to the one dimensional survival characteristics of its coordinates:

Theorem 4 *Let $X(t)$ be a two-dimensional Lévy process (1) with equal cumulative claims $S(t) = S_1(t) = S_2(t)$ given by an arbitrary Lévy process. If $x_2 > x_1, p_2 < p_1$, then the two-dimensional survival probabilities associated to the or/sim ruin problems (5), (7), are given by:*

$$\begin{aligned}
\overline{\psi}_{\mathrm{or}}(x_1, x_2) &= \int_0^\infty \overline{\psi}_1(\mathrm{d}z, T|x_1)\, \overline{\psi}_2(z), \\
\overline{\psi}_{\mathrm{sim}}(x_1, x_2) &= \int_0^\infty \overline{\psi}_2(\mathrm{d}z, T|x_2)\, \overline{\psi}_1(z),
\end{aligned}$$

where T is given in (36), $\overline{\psi}_i(\mathrm{d}z, T|x_i)$ in (38) and $\overline{\psi}_i(z) = P_z(I_i(\infty) \geq 0)$ are perpetual one-dimensional survival probabilities.

Proof: Recall that

$$\overline{\psi}_{\mathrm{or}}(x_1, x_2) = P_{(x_1,x_2)}(\min\{X_1(t), X_2(t)\} \geq 0 \text{ for all } t \geq 0).$$

Next, we note that, if $x_2 > x_1$, it holds that the minimum

$$\min\{X_1(t), X_2(t)\} = \min\{x_1 - x_2 + (p_1 - p_2)t, 0\} + X_2(t)$$

is equal to $X_1(t)$ for $t \leq T$ and $X_2(t)$ for $t > T$, where T was defined in (36). Applying subsequently the Markov property of X_2 at time T shows that

$$\begin{aligned}
\overline{\psi}_{\mathrm{or}}(x_1, x_2) &= P_{(x_1,x_2)}(X_1(t) \geq 0 \text{ for } t \leq T, X_2(t) \geq 0 \text{ for } t \geq T) \\
&= \int_0^\infty P_{x_1}(X_1(T) \in \mathrm{d}z, I_1(T) \geq 0)\, P_z(I_2(\infty) \geq 0).
\end{aligned}$$

Similarly, the probability $\overline{\psi}_{\mathrm{sim}}(x_1, x_2)$ that S stays below the barrier $b_{max}(t)$ can be seen to be equal to

$$\overline{\psi}_{\mathrm{sim}}(x_1, x_2) = \int_0^\infty P_{x_2}(X_2(T) \in \mathrm{d}z, I_2(T) \geq 0)\, P_z(I_1(\infty) \geq 0).$$

□

3.2 Particular cases of Theorem 4

Combining Theorem 4 with Proposition 1 yields an expression for $\overline{\psi}_{\mathrm{or}}(x_1, x_2)$ in terms of the one-dimensional distributions of S. In the case that S is a compound Poisson process the next result, whose proof can be found in Section 6, expresses the one-dimensional distributions of S as a series:

Corollary 2 *Suppose S is a compound Poisson process whose jump sizes σ_i have the pdf f. If $x_2 > x_1$, Theorem 4 holds true where $\overline{\psi}_i(\mathrm{d}z, T|x_i)$ is given by (32) with*

$$P_{x_i}(X(t) \in \mathrm{d}x) = \mathrm{e}^{-\lambda t}\delta_{x_i+p_it}(\mathrm{d}x) + p(t, x_i + p_it - x)\mathrm{d}x$$

and

$$p(t,z) = \mathrm{e}^{-\lambda t}\sum_{n=1}^{\infty}\frac{(\lambda t)^n}{n!}f^{\star n}(z) \tag{39}$$

is the density of $S(t)$.

If, in addition, the claims sizes σ_i follow a phase-type distribution $(\boldsymbol{\beta}, \boldsymbol{B})$, i.e. $P[\sigma > x] = \boldsymbol{\beta}\mathrm{e}^{\boldsymbol{B}x}\mathbf{1}$, the solution simplifies. Indeed, in this case the one-dimensional ruin probability may be written in a simple matrix exponential form:

$$\psi_i(x_i) = \boldsymbol{\eta}_i\mathrm{e}^{\boldsymbol{Q}_ix_i}\mathbf{1} \tag{40}$$

with $\boldsymbol{Q}_i = \boldsymbol{B} + \boldsymbol{b}\boldsymbol{\eta}_i$ and $\boldsymbol{\eta}_i = \frac{\lambda}{p_i}\boldsymbol{\beta}(-\boldsymbol{B})^{-1}$ (see for example (4) in Asmussen et al. (2002)). Combining this explicit formula (40) with Theorem 4 yields the following result:

Corollary 3 *Suppose S is a compound Poisson process with phase-type jumps $(\boldsymbol{\beta}, \boldsymbol{B})$. If $x_2 > x_1$, it holds that*

$$\psi_{\mathrm{or}}(x_1,x_2) = P_{x_1}(I_1(T) < 0) + \boldsymbol{\eta}_2\int_0^{\infty}\mathrm{e}^{\boldsymbol{Q}_2z}\overline{\psi}_1(\mathrm{d}z, T|x_1)\mathbf{1}, \tag{41}$$

$$\psi_{\mathrm{sim}}(x_1,x_2) = P_{x_2}(I_2(T) < 0) + \boldsymbol{\eta}_1\int_0^{\infty}\mathrm{e}^{\boldsymbol{Q}_1z}\overline{\psi}_2(\mathrm{d}z, T|x_2)\mathbf{1}, \tag{42}$$

where $\boldsymbol{Q}_i = \boldsymbol{B} + \boldsymbol{b}\boldsymbol{\eta}_i$ and $\boldsymbol{\eta}_i = \frac{\lambda}{p_i}\boldsymbol{\beta}(-\boldsymbol{B})^{-1}$.

In the special case of exponential claims σ_i with rate μ equation (41) can be developed further by employing the technique of change of measure and by applying the Markov property of X_i. Indeed, as a particular case of the phase-type relation (41), we see that

$$\psi_{\mathrm{or}}(x_1,x_2) = P_{x_1}(I_1(T) < 0) + C_2E_{x_1}[\mathrm{e}^{-\gamma_2X_1(T)}\mathbf{1}_{\{I_1(T)\geq 0\}}]. \tag{43}$$

By a change of measure and using that $-\gamma_2x_1 + \kappa_1(-\gamma_2)T = -\gamma_2x_2$ we find that the second term in (43) is equal to

$$C_2\mathrm{e}^{-\gamma_2x_1+\kappa_1(-\gamma_2)T}E_{x_1}[\Lambda^{(-\gamma_2)}(T)\mathbf{1}_{\{I_1(T)\geq 0\}}] = C_2\mathrm{e}^{-\gamma_2x_2}P_{x_1}^{(-\gamma_2)}(I_1(T)\geq 0).$$

The probability ψ_{sim} can be treated using similar arguments. In conclusion, the original two-dimensional ruin problems $\psi_{\mathrm{or}}/\psi_{\mathrm{sim}}/\psi_{\mathrm{both}}$ have been reduced to one-dimensional finite time ruin problems $\psi_i^{(c)}(x,t) = P_x^{(c)}[I_i(t) < 0]$, as follows:

Corollary 4 *Suppose S is a compound Poisson process with exponential jumps. If $x_2 > x_1$, it holds that*

$$\begin{aligned}\psi_{\text{sim}}(x_1, x_2) &= \psi_2(x_2, T) + \psi_1(x_1)\overline{\psi}_2^{(-\gamma_1)}(x_2, T),\\ \psi_{\text{or}}(x_1, x_2) &= \psi_1(x_1, T) + \psi_2(x_2)\overline{\psi}_1^{(-\gamma_2)}(x_1, T),\\ \psi_{\text{both}}(x_1, x_2) &= w_1(x_1, T) + \psi_2(x_2)\psi_1^{(-\gamma_2)}(x_1, T),\end{aligned}$$

where $w_1(x,t) = w_{\lambda,\mu,p_1}(x,t)$ is given by

$$w_{\lambda,\mu,p}(x,t) = P_x(I(\infty) < 0, I(t) \geq 0) = \psi(x) - \psi(x,t)$$

for $I(t)$ defined for $X(t) = pt - S(t)$.

Remark. Recall (see Example 1) that shifting the measure from P to $P^{(-\gamma_i)}$ for a Lévy risk processes with premium p and exponential claim sizes of intensity μ is equivalent to using the parameters $\tilde{\mu}_i := \mu_{-\gamma_i}, \tilde{\lambda}_i := \lambda_{-\gamma_i}$ under P. When

$$\rho > \rho^* := p_2^2/p_1$$

we find that the adjustment parameter of X_1 under $P^{(-\gamma_2)}$ is positive and equals to

$$\gamma = \gamma_3 = \frac{\mu}{p_2}(\rho - p_2^2/p_1), \tag{44}$$

in the opposite case, $\rho \leq \rho^*$, this coefficient is zero. Moreover, the asymptotic constant C_3 in the first case satisfies $C_3C_2 = \frac{p_2}{p_1}$. Similarly, we find that under $P^{(-\gamma_1)}$, the drift of X_2 is always negative, $\kappa_2^{(-\gamma_1)\prime}(0) = \kappa_2'(-\gamma_1) < 0$, so that the adjustment parameter of X_2 is always zero. Noticing that $\overline{\psi}_1^{(-\gamma_2)}(x_1, T) = w_1^{(-\gamma_2)}(x_1, T) + (1 - C_3\mathrm{e}^{-\gamma_3 x_1})I_{[\rho>\rho^*]}$ and combining with above Corollary leads to

$$\begin{aligned}\psi_{\text{sim}}(x_1, x_2) &= C_2\mathrm{e}^{-\gamma_2 x_2} + \omega_2(x_1, x_2),\\ \psi_{\text{or}}(x_1, x_2) &= C_1\mathrm{e}^{-\gamma_1 x_1} - \omega_1(x_1, x_2) + \mathrm{e}^{-\gamma_2 x_2}(C_2 - (p_2/p_1)\mathrm{e}^{-\gamma_3 x_1})_+,\\ \psi_{\text{both}}(x_1, x_2) &= \omega_1(x_1, x_2) + \mathrm{e}^{-\gamma_2 x_2}\max\{C_2, (p_2/p_1)\mathrm{e}^{-\gamma_3 x_1}\},\end{aligned}$$

where $x_+ = \max\{x, 0\}$, $\gamma_i = \mu - \lambda/p_i$ $(i = 1, 2)$, $C_i = \lambda/(\mu p_i)$ and

$$\omega_i(x_1, x_2) = C_1\mathrm{e}^{-\gamma_1 x_1}w_{\widetilde{\lambda}_1,\widetilde{\mu}_1,p_i}(x_i, T) - C_2\mathrm{e}^{-\gamma_2 x_2}w_{\widetilde{\lambda}_2,\widetilde{\mu}_2,p_i}(x_i, T). \tag{45}$$

An explicit formula for $w_{\lambda,\mu,p}(x,t) = P_x(t < \tau < \infty)$ can be extracted from the literature [Asmussen (1984), Knessl and Peters (1994) (with $p = 1$) and Pervozvansky (1998)] - see Appendix 6.

Remark. In view of the fact (Example 2) that the perpetual ruin probability of a Brownian motion with drift is given by an exponential, the representation in Corollary 4 remains valid if S is replaced by a Brownian motion. In this case $\psi(x,t)$ is given in terms of inverse Gaussian distributions.

4 Two dimensional Cramér asymptotics

We now turn to the asymptotics of the ruin probabilities $\psi_{\rm or}/\psi_{\rm sim}/\psi_{\rm both}(x_1,x_2)$ in the case that the initial reserves tend to infinity according to a ray x_1/x_2. Section 4.1 is devoted to asymptotics for general two-dimensional Lévy processes (implying the result for a Cramér-Lundberg process as in Theorem 1(a)). In Section 4.2 we derive asymptotics for the particular (degenerated) risk process given in the introduction and we consider in Section 4.3 the example of exponential jumps.

4.1 General asymptotics

Let $X = (X_1, X_2)$ now be a two-dimensional Lévy process and assume that X_1, X_2 do not have monotone paths. We will denote by

$$\kappa(\boldsymbol{\theta}) = \kappa(\theta_1,\theta_2) = \log E[\mathrm{e}^{\theta_1 X_1(1)+\theta_2 X_2(1)}]$$

the joint cumulant of $X = (X_1, X_2)$, by $\Theta = \{\boldsymbol{\theta} : \kappa(\boldsymbol{\theta}) < \infty\}$ the domain of κ and by Σ the Cramér set,

$$\Sigma = \{\boldsymbol{\theta} = (\theta_1,\theta_2) \in \Theta : \kappa(\theta_1,\theta_2) \leq 0\}.$$

Suppose that the Cramér assumption hold for X_1 and X_2, that is, there exist $\gamma_1, \gamma_2 > 0$ such that

$$\kappa(-\gamma_1, 0) = \kappa(0, -\gamma_2) = 0. \tag{46}$$

Moreover, assume the following condition is satisfied by the partial derivatives of κ:

$$\frac{\partial\kappa}{\partial u}(u,v)\bigg|_{(u,v)=(-\gamma_1,0)} + \frac{\partial\kappa}{\partial v}(u,v)\bigg|_{(u,v)=(0,-\gamma_2)} > -\infty. \tag{47}$$

By $(P_{(x,y)}, x, y \in \mathbb{R})$ we will denote the family measures under which $X(0) = (x, y)$ and, for $c = (c_1, c_2) \in \Theta$, $P^{(c)}$ denotes the measure with Radon-Nikodym derivative with respect to P given by

$$\frac{\mathrm{d}P^{(c)}}{\mathrm{d}P}\bigg|_{\mathcal{F}_t} = \exp(c_1 X_1(t) + c_2 X_2(t) - \kappa(c_1, c_2)t),$$

where $\mathcal{F}_t$ denotes the P-completed sigma-algebra generated by $(X_s, s \leq t)$. We shall also use the notation (4) – (10) for the different ruin times of interest in this setting.

Theorem 5 *Suppose that the Cramér assumptions (46) and (47) hold and let $a > 0$. Then, as $K \to \infty$,*

$$\psi_{\rm or}(aK, K) \sim C_2\mathrm{e}^{-\gamma_2 K} + C_1\mathrm{e}^{-\gamma_1 aK}, \tag{48}$$

$$\psi_{\rm both}(aK, K) = \mathrm{o}(C_2\mathrm{e}^{-\gamma_2 K} + C_1\mathrm{e}^{-\gamma_1 aK}), \tag{49}$$

where C_1 and C_2 are given in (17).

Proof: We start with a few estimates. On the one hand, it holds that

$$\begin{aligned}\psi_{\mathrm{or}}(aK,K) &= \psi_1(aK)+\psi_2(K)-\psi_{\mathrm{both}}(aK,K)\\ &\leq \psi_1(aK)+\psi_2(K),\end{aligned} \tag{50}$$

while, on the other hand,

$$\psi_{\mathrm{or}}(aK,K) \geq \max\{\psi_1(aK),\psi_2(K)\}. \tag{51}$$

Note that, in view of (50) and the Cramér-Lundberg asymptotics (28), the asymptotics (48) imply those of (49):

$$\lim_{K\to\infty}\frac{\psi_{\mathrm{both}}(aK,K)}{C_1\mathrm{e}^{-\gamma_1 aK}+C_2\mathrm{e}^{-\gamma_2 K}}=0.$$

The rest of the proof is therefore devoted to showing (48).

If $\gamma_1 a > \gamma_2$ [resp. $\gamma_1 a < \gamma_2$], it follows, in view of the Cramér-Lundberg asymptotics (28), that the lower bound (51) and upper bound (50) are of the same order of magnitude, $C_2\mathrm{e}^{-\gamma_2 K}$ [resp. $C_1\mathrm{e}^{-\gamma_1 aK}$], as $K\to\infty$. Thus (48) is valid if $\gamma_1 a \neq \gamma_2$.

Next we turn to the case $\gamma_1 a = \gamma_2$. In this case we have to show that

$$\psi_{\mathrm{or}}(aK,K) \sim C_1\mathrm{e}^{-\gamma_1 aK}+C_2\mathrm{e}^{-\gamma_2 K} \qquad \text{as } K\to\infty.$$

Noting that (with $\tau_1=\tau_1(aK)$ and $\tau_2=\tau_2(K)$) it holds that

$$\begin{aligned}\psi_{\mathrm{or}}(aK,K) &= P_{(aK,K)}(\tau_1\leq\tau_2,\tau_1<\infty)+P_{(aK,K)}(\tau_2\leq\tau_1,\tau_2<\infty)\\ &- P_{(aK,K)}(\tau_1=\tau_2<\infty),\end{aligned}$$

we shall show that (i) the first two terms are of the order $C_1\mathrm{e}^{-\gamma_1 aK}+C_2\mathrm{e}^{-\gamma_2 K}$ while (ii) the third term is of smaller order.

(i) For the first term of last display it follows, by a change of measure, that

$$\mathrm{e}^{-\gamma_1 aK}E^{(-\gamma_1,0)}_{(aK,K)}(\mathrm{e}^{-\gamma_1 X(\tau_1)}\mathbf{1}_{\{\tau_1\leq\tau_2,\tau_1<\infty\}}).$$

We claim that, as $K\to\infty$, it holds that

$$E^{(-\gamma_1,0)}_{(aK,K)}(\mathrm{e}^{-\gamma_1 X(\tau_1)}\mathbf{1}_{\{\tau_1\leq\tau_2,\tau_1<\infty\}})\to C_1. \tag{52}$$

To prove of this claim we compare the asymptotic behaviour of τ_1 and τ_2 as $K\to\infty$, following Glasserman and Wang (1997). If $E^{(-\gamma_1,0)}[X_2(1)]\leq 0$, then $P^{(-\gamma_1,0)}_{(aK,K)}(\tau_2<\infty)=1$ (note that $P^{(-\gamma_1,0)}_{(aK,K)}(\tau_1<\infty)=1$ since it holds that $E^{(-\gamma_1,0)}[X_1(1)]\leq 0$). Further, by spatial inhomogeneity (τ_1,τ_2) under $P^{(-\gamma_1,0)}_{(u_1,u_2)}$ has the same law as $(T_1(u_1),T_2(u_2))$ under $P^{(-\gamma_1,0)}_{(0,0)}$ (where $T_i(u_i)=\inf\{t\geq 0: X_i(t)\leq -u_i\}$). Since X has independent increments, it follows by a renewal argument that, as $K\to\infty$,

$$\frac{T_1(aK)}{T_2(K)}=a\frac{T_1(aK)}{aK}\frac{K}{T_2(K)}\to a\frac{E^{(\gamma_1,0)}[X_2(1)]}{E^{(\gamma_1,0)}[X_1(1)]}=a\frac{\frac{\partial\kappa}{\partial\theta_2}(-\gamma_1,0)}{\frac{\partial\kappa}{\partial\theta_1}(-\gamma_1,0)} \tag{53}$$

$P^{(-\gamma_1,0)}_{(0,0)}$-a.s. The convexity of Σ now implies that the right-hand side of (53) is bounded above by $\frac{a\gamma_1}{\gamma_2}$ (which is equal to 1 as $\gamma_2 = a\gamma_1$). Indeed, as Σ is convex it holds that

$$[(u,v) - (-\gamma_1, 0)] \cdot \nabla\kappa(-\gamma_1, 0) \leq 0$$

for all points $(u,v) \in \Sigma$ (where $\cdot$, denotes the inner-product) and the inequality follows by choosing $(u,v) = (0, -\gamma_2)$. If $E^{(-\gamma_1,0)}[X_2(1)] > 0$, then $P^{(-\gamma_1,0)}_K(\tau_2 = \infty) \to 1$ as $K \to \infty$. Therefore, the limits as K tends to infinity of the left-hand side of (52) and $E^{(-\gamma_1,0)}_{(aK,K)}(\mathrm{e}^{-\gamma_1 X(\tau_1)}\mathbf{1}_{\{\tau_1<\infty\}})$ coincide. In view of the Cramér-Lundberg asymptotics (28) the latter quantity converges to C_1 as $K \to \infty$ and the claim (52) follows.

The second term can be treated similarly to find that, as $K \to \infty$,

$$P_{(aK,K)}(\tau_2 \leq \tau_1, \tau_2 < \infty) \sim C_2 \mathrm{e}^{-\gamma_2 K}.$$

(ii) Note that the third term is dominated by $\psi_{\mathrm{sim}}(aK,K)$. Choose $\beta \in (0,1)$ and write $\gamma^\beta = \beta(\gamma_1, 0) + (1-\beta)(0,\gamma_2)$. By strict convexity of the set Σ there exists a $-\gamma^* \in \Sigma$ such that $\gamma^*_i > \gamma^\beta_i$, $(i = 1,2)$. By changing the measure, we see that $\psi_{\mathrm{sim}}(aK,K)$ is equal to

$$\mathrm{e}^{-(\gamma_1^* a+\gamma_2^*)K} E^{(-\gamma^*)}_{(aK,K)}[\mathrm{e}^{\gamma_2^* X_2(\tau_{\mathrm{sim}})+\gamma_1^* X_1(\tau_{\mathrm{sim}})+\kappa(-\gamma_1^*,-\gamma_2^*)\tau_{\mathrm{sim}}}\mathbf{1}_{\{\tau_{\mathrm{sim}}<\infty\}}].$$

Since $X_i(\tau_{\mathrm{sim}}) < 0$ and $\kappa(-\gamma_1^*, -\gamma_2^*) \leq 0$, the expectation in this display is bounded by 1. Therefore

$$\psi_{\mathrm{sim}}(aK,K) \leq \mathrm{e}^{-(\gamma_1^* a+\gamma_2^*)K} = \mathrm{o}(\mathrm{e}^{-(\gamma_1^\beta a+\gamma_2^\beta)K}) = \mathrm{o}(\mathrm{e}^{-\gamma_2 K}) = \mathrm{o}(\mathrm{e}^{-\gamma_1 aK}) \quad (54)$$

as $K \to \infty$ (recalling that $a\gamma_1 = \gamma_2$) and the proof is finished. □

4.2 Degenerate risk process asymptotics

To establish asymptotics for a risk-process with the particular structure described in the introduction. To be more precise, we restrict ourselves to a two-dimensional Lévy process (X_1, X_2) with $X_i(t) = p_i t - S(t)$, $i = 1,2$, for a spectrally positive Lévy process S and still assume that (15) holds true, writing κ_i for the cumulant of X_i. Note that the joint cumulant κ of (X_1, X_2) is in this case related to κ_i by

$$\kappa(\theta_1,\theta_2) = \kappa_1(\theta_1+\theta_2) - \theta_2(p_1 - p_2) = \kappa_2(\theta_1+\theta_2) + \theta_1(p_1-p_2).$$

From the theory of large deviations of first passage times it is well known that a central role in the description of the exponents is played by the support functional $\tilde{I}$ of the Cramér set, Σ, intersected with the positive quadrant:

$$\tilde{I}(\boldsymbol{a}) = \sup_{-\boldsymbol{\theta}\in\Sigma\cap\mathbb{R}^2_-} \langle \boldsymbol{a}, \boldsymbol{\theta}\rangle, \qquad \boldsymbol{a} \in \mathbb{R}^2_+, \quad (55)$$

where $\langle \cdot, \cdot \rangle$ denotes the standard inner-product in $\mathbb{R}^2$ and $\mathbb{R}^2_\pm = \{(x_1, x_2) : \pm x_i \le 0\}$. In this section we explicitly solve this variational problem and relate it to the asymptotics for $\psi_{\rm sim}$ and $\psi_{\rm both}$.

The solution of the variational problem (55) gives rise to a division of the upper cone $\mathcal{C}^c = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 > x_1 > 0\}$ into three open (possibly empty) sub-cones $\mathcal{D}_1$, $\mathcal{D}_2$ and $\mathcal{D}_0 = \mathcal{D}_1 \cup \overline{\mathcal{D}}_2^c$ defined in (18) and (19), where $\overline{\mathcal{D}}_2^c = \mathcal{C}^c \backslash \overline{\mathcal{D}}_2$ and $\overline{\mathcal{D}}_2$ denotes the closure of $\mathcal{D}_2$. The sub-cones $\mathcal{D}_i$, $i = 1, 2$, are distinct and contained in $\mathcal{C}^c$:

Lemma 2 *The following relation hold true for s_1, s_2 and γ_1, γ_2:*

(a) $s_2 < \gamma_2/\gamma_1 < s_1$; (b) $s_1 < 1$ and $s_2 > 0$ iff $\kappa_1'(-\gamma_2) < 0$; (c) $0 < \gamma_2 < \gamma_1$. In particular, $\mathcal{D}_1 \cup \mathcal{D}_2 \subset \mathcal{C}^c$, $\mathcal{D}_1 \cap \overline{\mathcal{D}}_2^c \ne \emptyset$ and $\mathcal{D}_2 \ne \emptyset$ iff $s_2 > 0$.

Proof of Lemma 2: (a,b) Writing

$$\frac{\kappa_1'(s)}{\kappa_2'(s)} = \frac{\kappa_2'(s) + p_1 - p_2}{\kappa_2'(s)} = 1 + \frac{p_1 - p_2}{\kappa_2'(s)},$$

it follows that $s_1 < 1$, since $\kappa_2'(-\gamma_1) < 0$, and that $s_2 < s_1$, since, by the strict convexity of κ_2, κ_1/κ_2 is strictly decreasing on the domain $\mathcal{D}$ of κ_i and $-\gamma_1 < -\gamma_2$. Finally, note that on the ray $x_1/x_2 = \gamma_2/\gamma_1$ it holds that

$$\frac{x_2}{T(x_1, x_2)} = \frac{p_1 - p_2}{1 - \gamma_2/\gamma_1} = \frac{\kappa_2(-\gamma_1) - \kappa_1(-\gamma_1)}{\gamma_1 - \gamma_2} = \frac{\kappa_2(-\gamma_1) - \kappa_2(-\gamma_2)}{\gamma_1 - \gamma_2}.$$

The strict convexity of κ_2 thus implies that along the ray $x_1/x_2 = \gamma_2/\gamma_1$ it holds that $-\kappa_2'(-\gamma_2) < x_2/T(x_1, x_2) < -\kappa_2'(-\gamma_1)$. In view of Lemma 4 below we deduce that $s_2 < \gamma_2/\gamma_1$. Similarly we prove that $\gamma_2/\gamma_1 < s_1$. Part (c) follows from (a) and (b). □

Denote by κ_1^* and κ_2^* the convex conjugates (34) and Θ_1 and Θ_2 for the domains of κ_1 and κ_2, respectively and note that, for $\boldsymbol{a} = (a_1, a_2)$ with $a_2 > 0$, $\tilde{I}(\boldsymbol{a}) = a_2 \tilde{I}(a_1/a_2, 1)$. The solution of the variational problem (55) now reads as follows:

Proposition 3 *Let $a \in (0, 1)$ with $a \ne s_1, s_2$ and suppose there exists a $\theta \in \operatorname{int}\Theta_1$ such that $\kappa_1'(-\theta) = -v_a^1$. It holds that*

$$\tilde{I}(a, 1) = \gamma_2 I_{(0,s_2)}(a) + \gamma(a) I_{(s_2,s_1)}(a) + a\gamma_1 I_{(s_1,1)}(a),$$

where $\gamma(a) := \kappa_2^*(v_a^2)/v_a^2 = a\kappa_1^*(v_a^1)/v_a^1$ and

$$v_a^1 = a(p_1 - p_2)/(1 - a), \quad v_a^2 = v_a^1/a = (p_1 - p_2)/(1 - a). \tag{56}$$

Further, $\gamma(a) > a\gamma_1$ and $\gamma(a) > \gamma_2$.

Proof of Proposition 3 The linear functional $(u, v) \mapsto -ua - v$ attains its maximum over the closed set $V = \Sigma \cap \mathbb{R}^2_-$ at $\partial\Sigma \cap \mathbb{R}^2_-$ (where $\partial\Sigma$ denotes the boundary of Σ). If the maximum is attained in the interior of $\partial\Sigma \cap \mathbb{R}^2_-$) the maximiser (u^*, v^*) satisfies $\kappa(u^*, v^*) = -v^*(p_1 - p_2) + \kappa_1(u^* + v^*) = 0$ and

$$\nabla\kappa(u^*, v^*) = (\kappa_1'(u^* + v^*), \kappa_2'(u^* + v^*)) = \kappa_2'(u^* + v^*)(a, 1).$$

Denoting by θ_a the root of $\kappa_1'(-s) = -v_a^1$ (which is also the root of $\kappa_2'(-s) = -v_a^2$) it follows that

$$u^* = -\kappa_2(-\theta_a)(p_1 - p_2) \qquad \text{and} \qquad v^* = \kappa_1(-\theta_a)/(p_1 - p_2). \tag{57}$$

In particular, $u^*a + v^* = \theta_a a + (1 - a)\kappa_1(-\theta_a)/(p_1 - p_2) = -a\kappa_1^*(v_a^1)/v_a^1$. If $a \in (s_2, s_1)$, it follows from Lemma 4 below that $\kappa_1(-\theta_a) < 0 < \kappa_2(-\theta_a)$ so that $u^* < 0$ and $v^* < 0$ and (u^*, v^*) is indeed the maximiser. However, if $a < s_2$ or $a > s_1$, $\kappa_1(-\theta_a)\kappa_2(-\theta_a) > 0$ and $(u^*, v^*) \notin \mathbb{R}^2_-$ and it can be directly verified that the maximum is attained at $(0, -\gamma_2)$ if $a < s_2$ and at $(-\gamma_1, 0)$ if $a > s_1$.

Finally, note that $\kappa_1^*(v) = \kappa_2^*(v+p_1-p_2)$. Moreover, $v_a^1 = a(p_1-p_2)/(1-a) = (p_1 - p_2)/(1 - a) - (p_1 - p_2) = v_a^2 - (p_1 - p_2)$. Also, from the definition of κ_i^* and strict convexity, we see that $\kappa_i^*(s) > -\gamma_i s$ for all $s \neq -\kappa_i'(-\gamma_i)$. $\square$

We are now ready to state the asymptotics of ψ_{both} and ψ_{sim} in the setting of this subsection.

Theorem 6 *Let $a \in (0, 1)$. Assume that the Cramér assumptions (46) hold true and there exists a $\theta \in \operatorname{int}\Theta_1$ such that $\kappa_1'(-\theta) = -v_a^1$.*

(i) If $(aK, K) \notin \mathcal{D}_0$, then, as $K \to \infty$,

$$\psi_{\text{sim}}(aK, K) \sim \psi_{\text{both}}(aK, K) \quad \sim \quad \begin{cases} C_1 \mathrm{e}^{-\gamma_1 aK} & (aK, K) \in \overline{\mathcal{D}}_1^c \\ C_2 \mathrm{e}^{-\gamma_2 K} & (aK, K) \in \mathcal{D}_2, \end{cases}$$

where the region $0 < a < s_2$ is understood to be empty if $s_2 = 0$.

(ii) If $(aK, K) \in \mathcal{D}_0$, then $\log\psi_{\text{sim}}(aK, K) \sim -\gamma(a)K$ as $K \to \infty$.

(iii) Suppose S is a compound Poisson process and let $s_2 > 0$. If $(aK, K) \in \mathcal{D}_0$, then $\log\psi_{\text{both}}(aK, K) \sim -\gamma(a)K$ as $K \to \infty$.

The proof of the Theorem is based on the following estimates that link ψ_{sim} and ψ_{both} to one-dimensional finite time ruin probabilities.

Lemma 3 *(i) $\psi_1(x_1) - \psi_1(x_1, T) + \psi_2(x_2, T) \leq \psi_{\text{sim}}(x_1, x_2) \leq \psi_{\text{both}}(x_1, x_2)$.*

(ii) $\psi_{\text{both}}(x_1, x_2) \leq \psi_1(x_1) - \psi_1(x_1, T) + \mathrm{e}^{-\gamma_2 x_2} P_{x_1}^{(-\gamma_2)}(\tau_1 < T)$.

Proof: By a change of measure, it follows that

$$\begin{aligned} \psi_{\text{both}}(x_1, x_2) &= \psi_1(x_1) - \psi_1(x_1, T) + \mathrm{e}^{-\gamma_2 x_2} E_{(x_1, x_2)}^{(-\gamma_2)}[\mathrm{e}^{\gamma_2 X_2(\tau_2)} \mathbf{1}_{\{\tau_1 < T < \tau_2 < \infty\}}] \\ &\leq \psi_1(x_1) - \psi_1(x_1, T) + \mathrm{e}^{-\gamma_2 x_2} P_{x_1}^{(-\gamma_2)}(\tau_1 < T). \end{aligned}$$

Further, for the first inequality, we note that

$$\psi_{\text{sim}}(x_1, x_2) = \psi_2(x_2, T) + P_{(x_1, x_2)}\left(\inf_{s<T} X_2(s) > 0, \inf_{T \leq s < \infty} X_1(s) < 0\right).$$

Since $\inf_{s<T} X_2(s) \geq \inf_{s<T} X_1(s)$ it follows that the second term in this display is bounded below by $\psi_1(x_1) - \psi_1(x_1, T)$. $\square$

Next write $T_i = x_i/[-\kappa_i'(-\gamma_i)]$, $\widetilde{T}_i = x_i/(-\kappa_i'(-\gamma_{3-i}))$ for a tilted version of T_i and recall $T(x_1, x_2)$ was defined in (36). The following result shows that it is equivalent to let the initial reserves x_1, x_2 tend to infinity while keeping x_1/x_2 constant or while keeping $x_i/T(x_1, x_2)$ constant, enabling us to link the asymptotics of the two-dimensional ruin problem to asymptotics of one-dimensional ruin probabilities:

Lemma 4 *(i)* $x_1/x_2 = a$ *iff* $x_i/T(x_1, x_2) = v_a^i$ $(i = 1, 2)$.
(ii) $T_i < T(x_1, x_2) \Leftrightarrow \widetilde{T}_{3-i} < T(x_1, x_2) \Leftrightarrow (x_1, x_2) \in \mathcal{D}_i$.

Proof: It is a matter of algebra to check that relation (i) follows by inserting the definition (36) of $T(x_1, x_2)$ and the expression for v_a^i. The relation (ii) follows then by using that $\kappa_1'(s) = p_1 - p_2 + \kappa_2'(s)$ and applying the first relation for $a = s_i$. $\square$

Remark. (Interpretation $\mathcal{D}_i$) The previous Lemma implies that we can give an alternative definition of the cones $\mathcal{D}_i$ as $\mathcal{D}_i = \{(x_1, x_2) \in \mathcal{C}^c : T_i < T(x_1, x_2)\}$. Noting that $T(x_1, x_2)$ is the first time that X_1 and X_2 are equal and that, in view of Lemma 1, $E_{x_i}^{(-\gamma_i)}[\tau_i] \sim T_i$ we thus deduce that $\mathcal{D}_i$ is the set of all rays $R_a := \{x_1 = ax_2\}$ such that

$$E_{x_i}^{(-\gamma_i)}[\tau_i] < T(x_1, x_2)$$

for $(x_1, x_2) \in \mathcal{R}_a$ large enough. Similarly, $\overline{\mathcal{D}_i}^c$ is the set of all rays such that $E_{x_i}^{(-\gamma_i)}[\tau_i] > T(x_1, x_2)$. for large enough initial reserves. The set $\mathcal{D}_i$ corresponds thus to the rays with expected time of ruin of company i before the reserves of both companies are equal.

Lemma 5 *Let* $a \in (0, 1)$. *The following hold true:*
(a) $a\kappa_1^{*(-\gamma_i)}(v_a^1)/v_a^1 = \kappa_2^{*(-\gamma_i)}(v_a^2)/v_a^2$.
(b) $\gamma_2 + \kappa_2^{*(-\gamma_2)}(v_a^2)/v_a^2 = \gamma_1 + a\kappa_1^{*(-\gamma_1)}(v_a^1)/v_a^1 = \gamma(a)$.

Proof: (a) follows from Proposition 3 and for (b) we note that $\kappa_2^{(-\gamma_2)}(s) = \kappa_2(s - \gamma_2)$ so that $\kappa_2^{*(-\gamma_2)}(v) = \sup_\beta(v\beta - \kappa_2(\beta - \gamma_2))$. $\square$

Proof of Theorem 6: The result for $\psi_{\text{sim}}(aK, K)$ follows by combining Lemmas 4 - 5 with the asymptotics given in Proposition 2 and Theorem 3. More specifically, if $x_1, x_2 \to \infty$ according to the ray $x_1/x_2 = a < s_2$ (or equivalently the ray $x_i/T(x_1, x_2) = v_a^i < -\kappa_i'(-\gamma_i)$) then $\psi_2(K, T) \sim C_2 \mathrm{e}^{-\gamma_2 K}$ and $\log(\psi_1(aK) - \psi_1(aK, T))$ is of the order $-a\kappa_1^*(v_a^1)K/v_a^1 = -\gamma(a)K < -\gamma_2 K$. In view of Lemma 5 it follows that the lower bound for $\psi_{\text{sim}}(aK, K)$ in Lemma 3 is equivalent to $C_2 \mathrm{e}^{-\gamma_2 K}$ as $K \to \infty$. Also, $\psi_{\text{sim}}(x_1, x_2) \leq \psi_2(K)$ which is equivalent to $C_2 \mathrm{e}^{-\gamma_2 K}$ as $K \to \infty$. Similarly, one can verify that in the case $a > s_1$ we have $\psi_{\text{sim}}(aK, K)$ and $\psi_{\text{both}}(aK, K)$ are both equivalent to $C_1 \mathrm{e}^{-\gamma_1 aK}$. In the intermediate area $a \in (s_2, s_1)$ we see that the logarithm of each of the terms of the lower bound is of the order $-a\kappa_1^*(v_a^1)K/v_a^1 = -\kappa_2^*(v_a^2)K/v_a^2 = -\gamma(a)K$. For the upper bound we use the inequality (54) giving $\psi_{\rm sim}(aK,K) \le \exp\{(au+v)K\}$ for $(u,v)=(u^*,v^*)$ defined in (57) and recall that $au^*+v^* = -\gamma(a)$.

Finally, we turn to an upper bound of $\psi_{\rm both}$ if $a\in(s_2,s_1)$. By Theorem 3 the first term of the lower bound in Lemma 3 is of the order $O(K^{-1/2}{\rm e}^{-\gamma(a)K})$ and the second term of the order $O(K^{-1/2}{\rm e}^{-\gamma(a)K})$ if $\kappa_1'(-\gamma_2)<0$ (using Lemma 2) and $O({\rm e}^{-(\gamma_2+\gamma_3)K})$ if $\kappa_1'(-\gamma_2)>0$ where $-\gamma_3>0$ solves $\kappa_1(-s-\gamma_2)=0$. We conclude that the lower and upper bound are of smaller order than $\min\{{\rm e}^{-\gamma_1 aK},{\rm e}^{-\gamma_2 K}\}$ as $K\to\infty$ (using Lemma 2) (where the case of $\kappa_1'(-\gamma_2)=0$ follows by adding a small drift to the process). Also, by comparing lower and upper bound we see that $\psi_{\rm both}(aK,K)\sim^{\log}\psi_{\rm sim}(aK,K)$ if $a\in(s_2,s_1)$ and $s_2>0$. □

4.3 Sharp asymptotics for exponential jumps

Restricting ourselves to the case that S is a compound Poisson process with exponential jumps, we can obtain explicit and more precise results, exploiting the explicit form of the ruin probabilities found in Corollary 4. It is a matter of calculus to verify from Example 1 that, for $i=1,2$, the vector of means is given by

$$(\kappa_1'(-\gamma_i),\kappa_2'(-\gamma_i)) = \left(p_1-\frac{p_i^2}{\rho},p_2-\frac{p_i^2}{\rho}\right).$$

From the previous section we know that the areas with different asymptotic behaviour of the ruin probabilities are separated by the rays

$$s_1=\frac{\frac{p_1^2}{\rho}-p_1}{\frac{p_1^2}{\rho}-p_2},\quad s_2=\frac{(\frac{p_2^2}{\rho}-p_1)_+}{\frac{p_2^2}{\rho}-p_2}.$$

Further, let γ_3 be the largest root of the equation $\kappa_1^{(-\gamma_2)}(-s)=0$ and set the corresponding ray, s_3, equal to

$$s_3=\frac{\kappa_1^{(-\gamma_2)\prime}(-\gamma_3)}{\kappa_2^{(-\gamma_2)\prime}(-\gamma_3)}=\frac{\kappa_1'(-\gamma_3-\gamma_2)}{\kappa_2'(-\gamma_3-\gamma_2)}.$$

The set $\mathcal{D}_3$ is defined as $\mathcal{D}_2$ but with s_2 replaced by s_3. Note that $s_2>0$ iff $\rho<\rho^*=p_2^2/p_1$ or equivalently $\kappa_1'(-\gamma_2)<0$. Note that in this case it holds that $s_2=s_3$ or $\mathcal{D}_2=\mathcal{D}_3$. In the case that $\rho>\rho^*$ we have that $s_2=0$, $s_3>0$, and γ_3 is given in (44), and the asymptotic behaviour of $\psi_{\rm both}(aK,K)$ as $K\to\infty$ differs according to whether or not (aK,K) lies in $\mathcal{D}_3$. Denote by $I_A=I_A(aK,K)$ the indicator of the set A which is one if $(aK,K)\in A$ and zero else and write $f(K)\approx g_1(K)+g_2(K)$ as $K\to\infty$ if $\lim_{K\to\infty}[f(K)-g_i(K)]/g_{3-i}(K)=1$.

Theorem 7 *Let $a \in (0, \infty)$. As $K \to \infty$ it holds that*

$$\begin{aligned}
\psi_{\rm or}(aK, K) &\approx C_1 e^{-\gamma_1 aK} I_{\mathcal{D}_1} + C_2 e^{-\gamma_2 K} I_{\overline{\mathcal{D}}_2^c} \\
&\quad + K^{-1/2} e^{-\gamma(a)K} (D_1(I_{\overline{\mathcal{D}}_1^c} - I_{\mathcal{D}_1}) + C_2 D_1^{(-\gamma_2)} (I_{\mathcal{D}_2} - I_{\overline{\mathcal{D}}_2^c})), \\
\psi_{\rm sim}(aK, K) &\approx C_1 e^{-\gamma_1 aK} I_{\overline{\mathcal{D}}_1^c} + C_2 e^{-\gamma_2 K} I_{\mathcal{D}_2} \\
&\quad + K^{-1/2} e^{-\gamma(a)K} (D_2(I_{\overline{\mathcal{D}}_2^c} - I_{\mathcal{D}_2}) + C_1 D_2^{(-\gamma_1)} (I_{\mathcal{D}_1} - I_{\overline{\mathcal{D}}_1^c})), \\
\psi_{\rm both}(aK, K) &\approx C_1 e^{-\gamma_1 aK} I_{\overline{\mathcal{D}}_1^c} + e^{-\gamma_2 K} \min\{C_2, (p_2/p_1) e^{-\gamma_3 aK}\} I_{\mathcal{D}_3} \\
&\quad + K^{-1/2} e^{-\gamma(a)K} (D_1(I_{\mathcal{D}_1} - I_{\overline{\mathcal{D}}_1^c}) + C_2 D_1^{(-\gamma_2)} (I_{\overline{\mathcal{D}}_3^c} - I_{\mathcal{D}_3})),
\end{aligned}$$

where $\gamma(a) = a\kappa_1^*(v_a^1)/v_a^1 = \kappa_2^*(v_a^2)/v_a^2$,

$$C_i = \frac{\rho}{p_i}, \qquad D_i = \frac{\theta_v - \theta_v'}{\theta_v |\theta_v'| \sqrt{2\pi \kappa_i''(-\theta_v)}} \sqrt{\frac{p_1 - p_2}{1 - a}}, \tag{58}$$

$D_i^{(-\gamma)}$ is D_i calculated for $P^{(-\gamma)}$ and $\theta_v' < 0$ is such that $\kappa_i(-\theta_v') = \kappa_i(-\theta_v)$ for $v = v_a^i$.

Proof: Recalling that $p_2 > \rho$ we can directly verify that the Cramér assumptions (15) are satisfied and that $\lim_{\theta \uparrow -\mu} \kappa_i'(\theta) = -\infty$, $i = 1, 2$. The result then follows by combining Corollary 4 with the finite time ruin asymptotics in Theorem 3, using Lemmas 4 and 5, as in the previous subsection. □

5 Proof of subexponential asymptotics

Throughout this section we shall assume that the distribution $F \in S^*$, that is the claim sizes σ are subexponential, and that $E\zeta^{2+\delta} < \infty$ for some $\delta > 0$. Let

$$I^{(i)}_{[x,y]}(w) = \int_x^y \overline{F}\,(w + m_i t)\ {\rm d}t$$

for $i = 1, 2$. Note that

$$H(aK, K) = I^{(2)}_{[0,T]}(K) + I^{(1)}_{[T,\infty]}(aK) \tag{59}$$

and

$$\frac{1}{m_2\mu}\overline{F}_I(K) = \frac{1}{\mu} I^{(2)}_{[0,\infty]}(K), \quad \frac{1}{m_1\mu}\overline{F}_I(aK) = \frac{1}{\mu} I^{(1)}_{[0,\infty]}(aK). \tag{60}$$

Straightforward calculations show that

$$I^{(1)}_{[T,\infty]}(aK) = (m_2/m_1) I^{(2)}_{[T,\infty]}(K). \tag{61}$$

We first prove asymptotics:

$$\lim_{K \to \infty} \frac{\psi_{\rm sim}(aK, K)}{H(aK, K)} = 1. \tag{62}$$

Let $T_0 = 0, T_n = \sum_{i=1}^n \zeta_i$ and $\Xi_0 = 0, \Xi_n = \sum_{i=1}^n (\sigma_i - E\sigma)$. From Fatou's lemma and Theorem 2 of Foss et al. (2005) we have

$$\limsup_{K\to\infty} \int \frac{P(\max_n(\Xi_n - g(n)) > K)}{\sum_{n=1}^\infty \overline{F}(K + g(n))} \; P(G \in \mathrm{d}g) \le 1 \; ,$$

where $G_i(n) = a_i K + p_i T_n - nE\sigma - K$ and $G(n) = \max_{i=1,2} G_i(n)$ are random discrete time processes on a possible function realisations $g : \mathbb{N} \to \mathbb{R}$ (with $a_1 = a, a_2 = 1$). Let $\mathcal{V} = \{T_n > -L + n(E\zeta - \epsilon) \quad \text{for all} \quad n\}$. Now,

$$\begin{aligned}
&\int \frac{P(\max_n(\Xi_n - g(n)) > K)}{\sum_{n=1}^\infty \overline{F}(K + g(n))} \; P(G \in \mathrm{d}g) \\
&\quad \ge \int \frac{P(\max_n(\Xi_n - g(n)) > K)}{\sum_{n=1}^\infty \overline{F}(K + g(n))} \mathbf{1}_{\{\mathcal{V}\}} \; P(G \in \mathrm{d}g) \\
&\quad \ge \frac{1}{\sum_{n=1}^\infty \overline{F}(\max_{i=1,2}(a_i K + p_i(E\zeta - \epsilon)n - nE\sigma))} \\
&\qquad \int P(\max_n(\Xi_n - g(n)) > K) \mathbf{1}_{\{\mathcal{V}\}} \; P(G \in \mathrm{d}g) \\
&\quad \ge \frac{1}{\sum_{n=1}^\infty \overline{F}(\max_{i=1,2}(a_i K + p_i(E\zeta - \epsilon)n - nE\sigma))} \\
&\qquad \Bigg(\int P(\max_n(\Xi_n - g(n)) > K) \; P(G \in \mathrm{d}g) \\
&\qquad\quad - \int P(\max_n(\Xi_n - g(n)) > K) \mathbf{1}_{\{\mathcal{V}^c\}} \; P(G \in \mathrm{d}g) \Bigg) . \qquad (63)
\end{aligned}$$

Note that $\psi_{\text{sim}}(aK, K) = \int P(\max_n(\Xi_n - g(n)) > K) \; P(G \in \mathrm{d}g)$ which is the first component in (63). We prove that

$$\lim_{\epsilon\to 0} \lim_{K\to\infty} \frac{\sum_{n=1}^\infty \overline{F}(\max_{i=1,2}(a_i K + p_i(E\zeta - \epsilon)n - nE\sigma))}{H(aK, K)} = 1. \qquad (64)$$

Indeed, note that $\sum_{n=1}^\infty \overline{F}(\max_{i=1,2}(a_i K + p_i(E\zeta - \epsilon)n - nE\sigma)) \sim \int_0^T \overline{F}(K + (m_2 - p_2\epsilon)t) \; dt + \int_T^\infty \overline{F}(aK + (m_1 - p_1\epsilon)t) \; \mathrm{d}t$. Now,

$$\int_0^T \overline{F}\left(K + (m_2 - p_2\epsilon)t\right) \; \mathrm{d}t = \frac{m_2}{m_2 - p_2\epsilon} \int_0^{\frac{m_2 - p_2\epsilon}{m_2} T} \overline{F}\left(K + m_2 t\right) \; \mathrm{d}t$$

which is bounded above by

$$\frac{m_2}{m_2 - p_2\epsilon} \int_0^T \overline{F}\left(K + m_2 t\right) \; \mathrm{d}t$$

and below by

$$\int_0^T \overline{F}\left(K + m_2 t\right) \; \mathrm{d}t.$$

Thus

$$\lim_{\epsilon\to 0}\lim_{K\to\infty}\frac{\int_0^T \overline{F}(K+(m_2-p_2\epsilon)t)\ \mathrm{d}t}{\int_0^T \overline{F}(K+m_2 t)\ \mathrm{d}t}=1.$$

Similarly, $\int_T^\infty \overline{F}(aK+(m_1-p_1\epsilon)t)\ \mathrm{d}t = \int_0^\infty \overline{F}(aK+(m_1-p_1\epsilon)t)\ \mathrm{d}t - \int_0^T \overline{F}(aK+(m_1-p_1\epsilon)t)\ \mathrm{d}t$ and

$$\lim_{\epsilon\to 0}\lim_{K\to\infty}\frac{\int_0^T \overline{F}(aK+(m_1-p_1\epsilon)t)\ \mathrm{d}t}{\int_0^T \overline{F}(aK+m_1 t)\ \mathrm{d}t}=1,$$

$$\lim_{\epsilon\to 0}\lim_{K\to\infty}\frac{\int_0^\infty \overline{F}(aK+(m_1-p_1\epsilon)t)\ \mathrm{d}t}{\int_0^\infty \overline{F}(aK+m_1 t)\ \mathrm{d}t}=1$$

since $\int_0^\infty \overline{F}(aK+(m_1-p_1\epsilon)t)\ \mathrm{d}t = \frac{m_1}{m_1-p_1\epsilon}\int_0^\infty \overline{F}(aK+m_1 t)\ \mathrm{d}t$. This proves (64). Moreover, second component in (63) is neglible with respect to $H(aK,K)$. Indeed,

$$\int P(\max_n(\Xi_n-g(n))>K)\mathbf{1}_{\{\mathcal{V}^c\}}\ P(G\in \mathrm{d}g)$$
$$\le \sum_{n=1}^\infty P(T_n<-L+n(E\zeta-\epsilon))P(\max_n(\Xi_n-nE\sigma)>K).$$

The first term goes to 0 as $L\to\infty$ by Fug and Nagaev (1971), Th. 2 and 3 (see Nagaev (1969) and Pinelis (1985)) and the assumption $E\zeta^{2+\delta}<\infty$. The second term is asymptotically equivalent to $\overline{F}_I(K)$ (see Embrechts and Veraverbeke (1982); Zachary (2004)) which is of order (up to constant) of $H(aK,K)$ (see (59) - (61)). Hence we proved that

$$\limsup_{K\to\infty}\frac{\psi_{\mathrm{sim}}(aK,K)}{H(aK,K)}\le 1. \tag{65}$$

We will prove now the lower bound of $\psi_{\mathrm{sim}}(aK,K)$. First note that taking interarrival times $\zeta\vee c$ for some $c>0$ instead of original ζ we decrease the ruin probability. Without loss of generality we can then assume that $\zeta>c$ for some $c>0$. Similarly as before, from Fatou's lemma and Theorem 2 of Foss et al. (2005) we have

$$1\le \liminf_{K\to\infty}\frac{1}{\sum_{n=1}^\infty \overline{F}(\max_{i=1,2}(a_iK+p_i(E\zeta+\epsilon)n-nE\sigma))}$$
$$\left(\psi_{\mathrm{sim}}(aK,K)-\int P(\max_n(\Xi_n-g(n))>K)\mathbf{1}_{\{\mathcal{U}^c\}}\ P(G\in \mathrm{d}g)\right),$$

where $\mathcal{U}=\{T_n<L+n(E\zeta+\epsilon)\quad \text{for all}\quad n\}$. We have

$$\int P(\max_n(\Xi_n-g(n))>K)\mathbf{1}_{\{\mathcal{U}^c\}}\ P(G\in \mathrm{d}g)$$
$$\le \sum_{n=1}^\infty P(T_n<L+n(E\zeta+\epsilon))P(\max_n(\Xi_n-nE\sigma)>K)$$

which is negligible with respect to $H(aK,K)$ when $K\to\infty$ and $L\to\infty$. In the same way as above we can prove that

$$\lim_{\epsilon\to 0}\lim_{K\to\infty}\frac{\sum_{n=1}^{\infty}\overline{F}(\max_{i=1,2}(a_iK+p_i(E\zeta+\epsilon)n-nE\sigma))}{H(aK,K)}=1.$$

Thus

$$\liminf_{K\to\infty}\frac{\psi_{\rm sim}(aK,K)}{H(aK,K)}\geq 1 \tag{66}$$

and in view of (65) we see that (62) holds.

In the same way (considering functional of minimum instead of maximum) we can prove that

$$\lim_{K\to\infty}\frac{\psi_{\rm or}(aK,K)}{\int_0^\infty\overline{F}\left(\min\left\{aK+m_1t,K+m_2t\right\}\right)\,{\rm d}t}=1 \tag{67}$$

which gives (22). Recall that $\psi_{\rm both}(aK,K)\geq\psi_{\rm sim}(aK,K)$ hence

$$\liminf_{K\to\infty}\frac{\psi_{\rm both}(aK,K)}{H(aK,K)}\geq 1.$$

We prove now the asymptotic upper bound. Note that

$$\psi_{\rm both}(aK,K)=P(\zeta_2(K)<\infty)-P(\zeta_2(K)<\infty,\zeta_1(aK)=\infty).$$

Appealing to Veraverbeke's Theorem (see Embrechts and Veraverbeke (1982); Zachary (2004) gives an alternative short proof for this result) we see that

$$P(\zeta_2(K)<\infty)=\psi_i(a_iK)\sim\frac{1}{m_2\mu}\overline{F}_I(K).$$

Hence for large K we have

$$P(\zeta_2(K)<\infty)\leq(1+\delta)\frac{1}{m_2\mu}\overline{F}_I(K) \tag{68}$$

for given $\delta>0$. Moreover, writing $M=\min\{n:T_n\geq T\}-1$ and $\overline{\Xi}_n$ for independent copy of Ξ_n starting from 0, for given $\epsilon>0$ and sufficiently large L

we have

$$
\begin{aligned}
P(\zeta_2(K) < \infty, \zeta_1(aK) = \infty) \geq \int & P(\max_{n>M}(\Xi_n - g_2(n)) > K, \\
& \max_{n\leq M}(\Xi_n - g_1(n)) < aK, \max_{n>M}(\Xi_n - g_1(n)) < aK) \\
& \mathbf{1}_{\{\mathcal{U}\cap\mathcal{V}\}}\, P(G_1 \in \mathrm{d}g_1, G_2 \in \mathrm{d}g_2) \\
\geq & \int P(\max_{n>M}(\Xi_n - g_2(n)) > K, \max_{n\leq M}(\Xi_n - g_1(n)) < aK) \\
& \mathbf{1}_{\{\mathcal{U}\cap\mathcal{V}\}}\, P(G_1 \in \mathrm{d}g_1, G_2 \in \mathrm{d}g_2) \\
& - \int P(\max_{n>M}(\Xi_n - g_1(n)) > aK)\mathbf{1}_{\{\mathcal{U}\cap\mathcal{V}\}}\, P(G_1 \in \mathrm{d}g_1) \\
\geq & \int P(\max_{n>M}(\Xi_n - g_2(n)) > K, \max_{n\leq M}(\Xi_n - g_1(n)) < aK, \Xi_M > -\epsilon M) \\
& \mathbf{1}_{\{\mathcal{U}\cap\mathcal{V}\}}\, P(G_1 \in \mathrm{d}g_1, G_2 \in \mathrm{d}g_2) \\
& - \int P(\max_{n>M}(\Xi_n - g_1(n)) > aK)\mathbf{1}_{\{\mathcal{U}\cap\mathcal{V}\}}\, P(G_1 \in \mathrm{d}g_1) \\
\geq & \int P(\max_{n>M}(\overline{\Xi}_{n-M} - g_2(n)) > K + \epsilon M) P(\max_{n\leq M}(\Xi_n - g_1(n)) < aK, \Xi_M > -\epsilon M) \\
& \mathbf{1}_{\{\mathcal{U}\cap\mathcal{V}\}}\, P(G_1 \in \mathrm{d}g_1, G_2 \in \mathrm{d}g_2) \\
& - \int P(\max_{n>M}(\Xi_n - g_1(n)) > aK)\mathbf{1}_{\{\mathcal{U}\cap\mathcal{V}\}}\, P(G_1 \in \mathrm{d}g_1)
\end{aligned}
$$

From Theorem 2 of Foss et al. (2005), for large K and given $\kappa > 0$ above expression could bounded below by:

$$
\begin{aligned}
& (1-\kappa) \int \sum_{n:T_n \geq T} \overline{F}\left(K + p_2 T_n - \delta_2 n\right) \mathbf{1}_{\{\mathcal{U}\cap\mathcal{V}\}}\, P(G_2 \in \mathrm{d}g_2) \\
& \quad - \int \sum_{n:T_n \geq T} \overline{F}\left(aK + p_1 T_n - \delta_1 n\right) \mathbf{1}_{\{\mathcal{U}\cap\mathcal{V}\}}\, P(G_1 \in \mathrm{d}g_1) \\
& \geq (1-2\kappa) \left(1 - \sum_{n=1}^{\infty} P(T_n < -L + n(E\zeta - \epsilon)) - \sum_{n=1}^{\infty} P(T_n > L + n(E\zeta + \epsilon))\right) \\
& \left(\int_T^{\infty} \overline{F}\left(K + m_2 t\right)\ \mathrm{d}t - \int_T^{\infty} \overline{F}\left(aK + m_1 t\right)\ \mathrm{d}t\right).
\end{aligned}
$$

giving for sufficiently large L

$$
(1-\delta) \left(\int_T^{\infty} \overline{F}\left(K + m_2 t\right)\ \mathrm{d}t - \int_T^{\infty} \overline{F}\left(aK + m_1 t\right)\ \mathrm{d}t\right). \tag{69}
$$

From (59), (68) and (69) we have

$$
\begin{aligned}
\psi_{\text{both}}(aK, K) &\leq (1+\delta) I^{(2)}_{[0,\infty)}(K) - (1-\delta)\left(I^{(2)}_{[T,\infty)}(K) - I^{(1)}_{[T,\infty)}(aK)\right) \\
&\leq H(aK, K) + \delta\left(2 I^{(2)}_{[T,\infty)}(K) + I^{(2)}_{[0,T]}(K) - I^{(1)}_{[T,\infty)}(aK)\right).
\end{aligned}
$$

Note that expression in brackets is of order (up to constant) of $H(aK, K)$ as $K \to \infty$ (see (59) and (61)). Taking then first $K \to \infty$ and then $\delta \to 0$ we derive

$$\limsup_{K\to\infty} \frac{\psi_{\text{both}}(aK, K)}{H(aK, K)} \leq 1$$

which completes the proof. □

6 Appendix

Lemma 1 (Explicit ruin) *The inverse Laplace transform of* $\overline{\Psi}^{(q)}(x)$ *in (29) is given by*

$$\overline{\psi}(x, t) = 1 - \psi(x, t) = [1 - C\mathrm{e}^{-\gamma x}]I_{(\gamma>0)} + w(x, t),$$

where $\gamma = \mu - \lambda/p$, $C = \frac{\lambda}{\mu p}$, *and*

$$w(x, t) = \frac{1}{\pi}\sqrt{\frac{\lambda}{\mu p}} \int_{s_-}^{s_+} \mathrm{e}^{a(q)x-qt} \sin(b(q)x - \phi(q)) \frac{\mathrm{d}q}{q} \tag{70}$$

where $s_\pm = (\sqrt{\lambda} \pm \sqrt{\mu p})^2$ *and*

$$a(q) = \frac{\lambda - \mu p - q}{2p}, \quad b(q) = \frac{\sqrt{4pq\mu - (\lambda - \mu p - q)^2}}{2p}, \tag{71}$$

$$\phi(q) = \arccos\left(\frac{p\mu + \lambda - q}{2\sqrt{\lambda\mu p}}\right). \tag{72}$$

Proof of Corollary 2: The transition probability $P(X_1(t) \in \mathrm{d}z)$ of X_1 can be found explicitly by conditioning on the number of jumps of the compound Poisson process S up till time t:

$$P(X_1(t) \in \mathrm{d}z) = \sum_{n=0}^{\infty} \mathrm{e}^{-\lambda t} \frac{(\lambda t)^n}{n!} F^{\star n}(\mathrm{d}(tp_1 - z)), \tag{73}$$

where $F^{\star n}(\mathrm{d}x)$ is the n-fold convolution of $F(\mathrm{d}x)$ and $F^{\star 0} = \delta_0$, the delta in zero. If the jump-size σ is a continuous random variable the only atom of $P(X_1(t) \in \mathrm{d}z)$ occurs in the absence of jumps, that is,

$$P(X_1(t) \in \mathrm{d}z) = \mathrm{e}^{-\lambda t}\delta_0(\mathrm{d}(tp_1 - z)) + p_1(t, z)\mathrm{d}z,$$

where $p_1(t, z) = \mathrm{e}^{-\lambda t} \sum_{n=1}^{\infty} \frac{(\lambda t)^n}{n!} f^{\star n}(tp_1 - z)$ with f is the probability density function of σ. The assertion now follows by noting that in this case the identity in (76) remains valid when we take instead of $P(X_1(t) \in \mathrm{d}z)$ the measure $p_1(t, z)\mathrm{d}z$ (as the atom only affects one t and we are integrating over t). □

6.1 Proofs of Section 2

Proof of Proposition 1: Write Q_t and $\widehat{Q}_t$ for the semi-groups corresponding to X (resp. $\widehat{X} = -X$) killed upon entering the negative half-axis $(-\infty, 0)$. By Hunt's switching identity (e.g. Bertoin (1996), Thm II.1.5) it holds for nonnegative measurable functions f, g that

$$\int_{\mathbb{R}} Q_t f(x) g(x) \mathrm{d}x = \int_{\mathbb{R}} f(x) \widehat{Q}_t g(x) \mathrm{d}x. \tag{74}$$

Moreover,

$$\begin{aligned}
\int_{\mathbb{R}} f(x)\widehat{Q}_t g(x)\mathrm{d}x &= \int_{\mathbb{R}} f(x) E_{-x}[g(-X(t))\mathbf{1}_{\{t<T(0)\}}]\mathrm{d}x \\
&= \int_{\mathbb{R}} E_x[f(X(t))]g(x)\mathrm{d}x - \int_{\mathbb{R}} f(x)\int_0^t E_0[g(-X(t-s))]P_{-x}[T(0)\in \mathrm{d}s] \\
&= \int_{\mathbb{R}} E_x[f(X(t))]g(x)\mathrm{d}x - \int_{\mathbb{R}} f(x)\int_0^t E_0[g(-X(t-s))]P[T(x)\in \mathrm{d}s],
\end{aligned}$$

where in the third line we used duality (e.g. Bertoin (1996) Prop. II.1.1), the strong Markov property and the fact that X is spectrally negative (and thus $X(T(0)) = 0$). Combining the last line and (74) with Kendall's identity, which is valid for spectrally negative Lévy processes (e.g. Bertoin (1996), Cor. VII.3, or Borovkov and Burq (2001)) and relates the distributions of $X(t)$ and the passage time $T(x)$:

$$tP(T(z)\in \mathrm{d}t)\mathrm{d}z = zP(X(t)\in \mathrm{d}z)\mathrm{d}t, \tag{75}$$

shows that the following equality between measures holds true:

$$\begin{aligned}
P_x(X(t)\in \mathrm{d}z, \inf_{0<s\le t} X(s)\ge 0)\mathrm{d}x &= P_x(X(t)\in \mathrm{d}z)\mathrm{d}x \\
&- z\int_0^t \frac{1}{s}P(X(s)\in \mathrm{d}z)P(-X(t-s)\in \mathrm{d}x)\mathrm{d}s, \\
&= P_x(X(t)\in \mathrm{d}z)\mathrm{d}x \\
&- z\int_0^t \frac{1}{(t-s)}P(X(t-s)\in \mathrm{d}z)P(-X(s)\in \mathrm{d}x)\mathrm{d}s,
\end{aligned} \tag{76}$$

so that, under the assumption (AC), the stated result follows. □

Proof of Corollary 1: Write $e(q)$ for an independent exponential time with mean q^{-1}. Taking the Laplace transform in t of (32) yields

$$\begin{aligned}
P_x(X(e(q))\in \mathrm{d}z,\ I(e(q))\ge 0) &= P_x(X(e(q))\in \mathrm{d}z) - u^q(-x)E[\mathrm{e}^{-qT(z)}]\mathrm{d}z \\
&= u^q(z-x) - u^q(-x)E[\mathrm{e}^{-qT(z)}]\mathrm{d}z,
\end{aligned} \tag{77}$$

where $u^q(x) = \frac{P(X(e(q))\in \mathrm{d}x)}{\mathrm{d}x}$ is a version of the potential density of X. From the fluctuation theory of spectrally negative Lévy processes it is well known (see e.g. Bingham (1975)) that $E[\mathrm{e}^{-qT(z)}] = \mathrm{e}^{-\Phi(q)z}$ for $z > 0$ and that $u^q(y) = \Phi'(q)\mathrm{e}^{-\Phi(q)y} - W^{(q)}(-y)\mathbf{1}_{y<0}$ (see Pistorius (2004) or Bingham(1975) for a proof). Inserting these expressions into (77) and subsequently taking the limit $x \downarrow 0$ shows that

$$P_0(X(e(q)) \in \mathrm{d}z, I(e(q)) \geq 0) = W^{(q)}(0^+)\mathrm{e}^{-\Phi(q)z}\mathrm{d}z = p^{-1}\mathrm{e}^{-\Phi(q)z}\mathrm{d}z,$$

where the second equality follows from a Tauberian theorem and the form of the exponent κ in this case. In view of the form of the Laplace transform of $T(z)$ and Kendall's identity (75) the proof is done. □

Acknowledgements

We are much indebted to Tomasz Rolski for his continuing advice on this paper. FA an MP gratefully acknowledge support from the London Mathematical Society, grant # 4416. FA and ZP acknowledge support by POLONIUM no 09158SD. ZP acknowledges support by KBN 1P03A03128 and NWO 613.000.310. MP acknowledges support by the Nuffield Foundation, grant no. NUF/NAL/000761/G.

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