Phase-type approximations to finite-time ruin probabilities in the Sparre-Andersen and stationary renewal risk models

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January 20, 2005

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Abstract

The present paper extends the "Erlangization" idea introduced by Asmussen, Avram, and Usabel (2002) to the Sparre-Andersen and stationary renewal risk models. Erlangization yields an asymptotically-exact method for calculating finite time ruin probabilities with phase-type claim amounts. The method is based on finding the probability of ruin prior to a phase-type random horizon, independent of the risk process. When the horizon follows an Erlang-l distribution, the method provides a sequence of approximations that converges to the true finite-time ruin probability as l increases. Furthermore, the random horizon is easier to work with, so that very accurate probabilities of ruin are obtained with comparatively little computational effort. An additional section determines the phase-type form of the deficit at ruin in both models. Our work exploits the relationship to fluid queues to provide effective computational algorithms for the determination of these quantities, as demonstrated by the numerical examples.

Keywords: Sparre-Andersen model, ladder height, maximal aggregate loss, deficit at ruin, phase-type distribution.

1. Introduction

The present paper is concerned with the determination of the probability of ruin in finite time in the Sparre-Andersen risk model and the stationary renewal risk model (also known as the stationary Sparre-Andersen model).

Traditionally, the exact determination of finite time ruin probabilities, in both the classical and the Sparre-Andersen risk models, has required the solution of rather complicated integro-differential equations. Explicit formulae for the probability of ruin exist for a limited number of cases, such as the classical model with exponential claim sizes. Even then, the form of the solution entails the evaluation of Bessel functions (see, for instance, Drekic & Willmot (2003)).

A popular alternative to working in continuous time in the classical model was introduced by Dickson & Waters (1991), who discretise time, and develop a recursion at successive time instants. As a result, one is able to avoid the solution of integro-differential equations. Stanford & Stroiński (1994) pursued a recursive approach, without needing to discretise, for the classical model, and this was extended for a limited selection of Sparre-Andersen models in Stanford, Stroiński, & Lee (2000). However, these two papers embed their recursions at claim epochs, thereby obtaining a sequence of exact probabilities of being ruined prior to the nth claim instant — a random quantity — rather than by time t. The main problem with this approach is that one cannot infer finite-time ruin probabilities, since claim occurrence times are correlated with the likelihood of ruin.

This difficulty was circumvented by Avram & Usabel (2003), who introduced the concept of ruin prior to an *independent* exponentially-distributed random horizon. This idea was extended in Asmussen, Avram & Usabel (2002) to the case of ruin before an independent phase-distributed horizon. A sequence of asymptotically-exact approximations is obtained when the horizon is Erlang-l distributed, as the degree l approaches infinity. As the numerical examples therein illustrate, a very reasonable approximation to the finite-time ruin probability is obtained, even for low order Erlang distributions, and this is further enhanced by resorting to a Richardson extrapolation.

By contrast, no "accurate yet efficient" method for calculating finite-time ruin probabilities currently exists in the Sparre-Andersen model. To our knowledge, there are only two lines of thought that present accurate finite-time ruin probabilities for the continuous-time Sparre Andersen model. Wikstad (1971) and Thorin & Wikstad (1973, 1977) use an advanced root-finding approach to determine finite time ruin probabilities, while Asmussen & Højgaard (1999) present a diffusion approximation for the finite-time ruin probability. Both of these are substantially more complicated than the Erlangization approach presented here. Furthermore, our results have the added advantage of being asymptotically exact.

The present work uses a probabilistic approach to analyze the structure of the ruin probability in the Sparre-Andersen model with phase-type claims, and exploits links to previous results due to Asmussen (2000) as well as results due to da Silva Soares & Latouche (2002) in the field of fluid queues. It is this latter approach that leads to our most efficient computational algorithms for the ruin probability.

The next four sections develop our probabilistic approach, following which corollaries relating to the deficit upon ruin are given. Numerical examples are given in section 7 to illustrate the method's accuracy. We conclude with comments on computational issues.

2. Initial Formulation of the Ruin Probability Prior to a Phase-Type Horizon

Inter-claim times (ICTs) $\{Y_i;\ i = 1, 2, \ldots\}$ in the Sparre-Andersen risk model constitute an independent and identically distributed (iid) renewal process with distribution function $A(t) = Pr\{Y_i \leq t\}$. Claim amounts $\{X_i;\ i = 1, 2, \ldots\}$ are iid random variables with a distribution of "phase type with representation (β, B)", meaning that $P(x) = Pr\{X_i \leq x\} = 1 - \beta \exp(Bx)\, e$, where $\exp(.)$ denotes the matrix exponential, and e a column vector of ones.

Claim amounts are independent of the inter-claim times.

A "relative security loading" $\theta > 0$ is charged on the expected payout per unit time, so premiums are accrued linearly over time at rate $c = (1 + \theta)E\{X\}/E\{Y\}$. The insurer's surplus at time t is $U_t = u + ct - \sum_{i=1}^{N_t} X_i$, where $u \geq 0$ is the initial surplus. The time of ruin is $\tau = \inf\{t : U_t < 0\}$, or $\tau = \infty$ if $U_t \geq 0$ for all $t \geq 0$. By rescaling time so that $c = 1$, we can also interpret $A(t)$ as the probability that the "inter-claim revenue" does not exceed t, consistent with the terminology in Asmussen (2000). The probability of ultimate ruin is $\psi(u) = Pr\{\tau < \infty\}$, and the finite-time ruin probability is $\psi(u, T) = Pr\{\tau < T\}$.

When claim amounts $X_i \sim PH_m(\beta, B)$, the ruin probability for a wide variety of models takes the form

$$\psi(u) \;=\; \eta \exp\{(B + b\eta)\, u\}\; e \tag{2.1}$$

where $b = -Be$ and the vector η is a defective probability vector. η takes various forms, dependent on the model. For the classical model, see for example Asmussen and Rolski (1991) and Avram and Usabel (2003). For the Sparre-Andersen model we consider here, Asmussen (2000) has shown that the maximal aggregate loss L also satisfies (**??**), and the stationary maximal aggregate loss L^e is closely related; see pp. 227-231. The solution for η is the result of a fixed-point calculation, and the line of thought in its determination plays a key role in our extension of Asmussen et al (2002). The fixed-point formula for the Sparre-Andersen case is

$$\eta = \varphi(\eta) = \beta \int_0^\infty \exp\{t(B + b\eta)\} dA(t) \tag{2.2}$$

and its probabilistic reasoning is as follows. Conditioning on the amount of revenue t earned during the inter-claim time, we seek the state of the underlying transient Markov chain at an "upcrossing" of the current level; i.e., the point where the maximal aggregate loss exceeds t (if indeed it ever does). The generator for transitions among the transient states is $(B + b\eta)$. This allows both for direct transitions during a single claim via the matrix B, and indirect ones when a claim ends but a subsequent ladder height restarts the process; the rates of these indirect transitions are given in the matrix $b\eta$. The resulting probability vector at the up-crossing, conditioned on t, is $\beta \exp\{t(B + b\eta)\}$. Removing the conditioning on t yields (**??**).

We turn now to a brief review of Asmussen et al (2002), whose development of the Erlangization approach for the classical risk model forms the starting point for our extension. Assume that the horizon is phase-type (ν, H) of dimension l, with $h = -He$ denoting the rates of absorption. Asmussen et al (2002) establish that the form of the ruin probability prior to a phase-distributed horizon $\psi(u, H)$ is given by

$$\psi(u, H) \;=\; \nu\eta\exp\{Uu\}e \tag{2.3}$$

where η is now a *matrix* of probabilities of size $l \times lm$, and

$$U = I_l \otimes B + (I_l \otimes b)\eta, \tag{2.4}$$

where "$\otimes$" refers to the Kronecker product. The matrix η represents the "up-crossing probabilities" (see Asmussen et al (2002)) that, starting from each horizon phase, the process will be in the various combinations of horizon phase and claim state at the next up-crossing of the current level.

In what follows, we establish that (**??**) is still valid in the Sparre-Andersen case. However, the approach in Asmussen et al (2002) for determining η is based on the solution of a Riccati equation, which cannot be used in the fuller Sparre-Andersen context. We resort to an extension of (**??**) instead.

A key element employed by Asmussen et al (2002) is the transformation of their original Lévy process for the aggregate loss $\tilde{X}(t) = ct - U_t$ into an equivalent Markovian fluid model $\{(X_t, \phi_t)\}$, which tracks the phase ϕ_t of the claim and the horizon. Equivalently, we track the surplus process U_t and ϕ_t, with fluid counterparts $\{(\Phi_t, \phi_t)\}$. Between claims, the fluid model and the original Lévy process coincide. Claims in the Lévy process become segments of linear decrease with slope -1 in the fluid model, resulting in added segments of artificial "time". Consequently, the phase of the horizon must be "frozen" during such segments, resulting in a larger dimension of the phase component ϕ_t of the fluid model during artificial time segments. The relationships between the surplus process and the fluid model are illustrated in Figure 1.

1. Between claims, the state tracks the phase of the horizon, with the transition probabilities of the phases after time t given by $\exp\{Ht\}$.

2. When a claim occurs, the horizon phase is "frozen", and the claim state evolves until the claim is fully paid. Mathematically, this enlargement

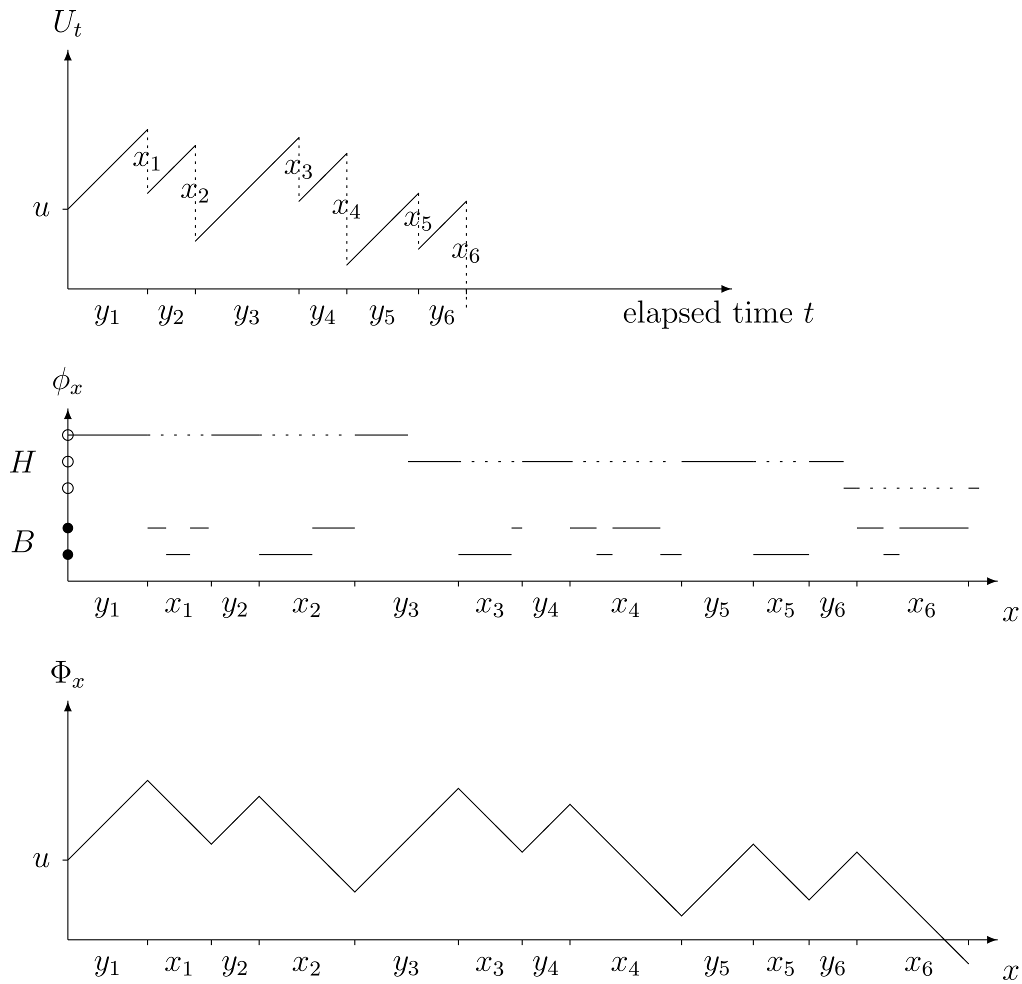

Figure 1: Relationship of the phase, fluid queue and risk process. The intervals between claims are y_1, y_2, ... The claim sizes are x_1, x_2, ... The claim size distribution has two phases, the horizon has three phases. The phase of the horizon is frozen when the phase of a claim evolves. In this case, ruin occurs before the end of the horizon.

of ϕ_t is described by the matrix

$$I_l \otimes \beta = \begin{bmatrix} \beta & 0 & 0 & \dots & 0 \\ 0 & \beta & 0 & \dots & 0 \\ 0 & 0 & \beta & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \beta \end{bmatrix}.$$

3. During artificial time segments, Asmussen et al (2002) show that the phase ϕ_t evolves according to a generator U given by (**??**).

The Sparre-Andersen case: While the Sparre-Andersen model no longer constitutes a Lévy process, one readily sees the equivalence between surplus process and fluid model shown by Figure 1 is still valid, and the phase ϕ_t still evolves in artificial time with generator U according to (**??**). What changes is the equation for η, which is obtained similarly to (**??**) by conditioning on the inter-claim time t, yielding the following:

$$\eta = \varphi(\eta) = \int_0^\infty \exp\{Ht\}\ (I_l \otimes \beta)\ \exp\{Ut\} dA(t). \tag{2.5}$$

When the horizon's phase is Erlang-distributed, $\nu = (1, 0, \dots, 0)$, and the matrix η takes an upper triangular, block Toeplitz form:

$$\eta = \begin{bmatrix} \eta_0 & \eta_1 & \eta_2 & \dots & \eta_{l-1} \\ 0 & \eta_0 & \eta_1 & \dots & \eta_{l-2} \\ 0 & 0 & \eta_0 & \dots & \eta_{l-3} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \eta_0 \end{bmatrix}$$

where η_i is a m-vector. Furthermore, $\exp\{Ht\}$ is an upper triangular $l \times l$ Toeplitz matrix with $e^{-at}(at)^n/n!$ on the nth super-diagonal, where a^{-1} is the mean duration per horizon stage.

In the next section, an alternative explicit expression is obtained, which enables us to exploit a result in Asmussen (2000) for the matrix moment generating function of a square matrix. Equation (**??**) is developed along different lines in section 4, to yield our most efficient computational algorithm for the probability of ruin prior to a phase-distributed horizon. This algorithm exploits previous results obtained in the field of fluid queues (see, for instance, da Silva Soares & Latouche (2002)).

3. An Alternative Embedding Approach

Rather than expanding the state space when the claim occurs, one could select the initial claim phase at the beginning, and "freeze" it during the inter-claim time. In this case, $\exp\{Ht\}$ is replaced by $\exp\{(H \otimes I_m)t\}$, yielding

$$\begin{aligned}
\varphi(\eta) &= (I_l \otimes \beta) \int_0^\infty \exp\{(H \otimes I_m)t\}\exp\{Ut\}dA(t) \\
&= (I_l \otimes \beta) \int_0^\infty \exp\{(H \otimes I_m + U)t\}dA(t) \qquad (3.1) \\
&= (I_l \otimes \beta) \int_0^\infty \exp\{(H \oplus B + (I_l \otimes b)\eta)t\}dA(t).
\end{aligned}$$

where the Kronecker sum operator "$\oplus$" is defined for square matrices A and B of dimensions c and d respectively by $A \oplus B = A \otimes I_d + I_c \otimes B$. We note that the foregoing result is **only true** if the matrix $(H \otimes I_m)$ commutes with the matrix $U = I_l \otimes B + (I_l \otimes b)\eta$, which we prove in the lemma below. But first we note the immediate corollary of the foregoing:

Corollary 1: The matrix η can also be obtained as the solution to

$$\begin{aligned} \eta &= I_l \otimes \beta \sum_{n=0}^{\infty} (H \otimes I_m + U)^n M_n / n! \\ &= I_l \otimes \beta \sum_{n=0}^{\infty} (H \oplus B + (I_l \otimes b)\eta)^n M_n / n! \end{aligned} \tag{3.2}$$

whenever the infinite sums converge (where M_n denotes the nth moment of the inter-claim time distribution).

Proof: Follows directly from expansion of the matrix exponentials.

Remark: We note in particular that the sum does not converge for log-normal inter-claim times, despite the existence of all the moments M_n.

Lemma 2: The matrices $(H \otimes I_m)$ and U commute.

Proof: Observe that $(I_l \otimes b)\eta = (I_l \otimes b\beta)G$, where

$$G = \int_0^{\infty} \exp\{(H \otimes I_m)t\} \exp\{(I_l \otimes B + (I_l \otimes b\beta)G)t\} dA(t). \tag{3.3}$$

Therefore, it is sufficient for us to show that $(H \otimes I_m)$ commutes with $U = I_l \otimes B + (I_l \otimes b\beta)G$. The implicit form of (**??**) suggests a sequence of matrices

$$G_{k+1} = \int_0^{\infty} \exp\{(H \otimes I_m)t\} \exp\{(I_l \otimes B + (I_l \otimes b\beta)G_k)t\} dA(t). \tag{3.4}$$

starting from $G_0 = 0$. It is readily observed that the sequence of matrices $\{G_k\}$ is monotonically non-decreasing entry-wise, and each is bounded above by G, so that there exists a limit to the sequence.

We now show the sequence converges to G itself. Observe that

$$G_1 = \int_0^{\infty} \exp\{(H \otimes I_m)t\} \exp\{(I_l \otimes B)t\} dA(t) \tag{3.5}$$

represents the matrix of probabilities that the horizon is not reached before the next claim, and the claim amount exceeds the revenue earned during the inter-claim time. Similarly,

$$G_2 = \int_0^\infty \exp\{(H \otimes I_m)t\}\exp\{(I_l \otimes B + (I_l \otimes b\beta)G_1)t\}dA(t) \tag{3.6}$$

represents the matrix of probabilities that the horizon is not reached before the next claim, and *either* a) the claim amount exceeds the revenue earned during the inter-claim time, *or* b) during the *subsequent* inter-claim time, the horizon still is not reached, and that claim amount exceeds the revenue earned during that interval. Continuing in this way, the sequence of matrices on both sides converges to the true matrix G, yielding (**??**).

Furthermore, $(H \otimes I_m)$ commutes with each matrix in the sequence, and consequently with the limit of the sequence, which is G. Therefore necessarily it commutes with $U = (I_l \otimes B) + (I_l \otimes b\beta)G$, which completes the proof.

Corollary 3: Suppose that inter-claim times are phase-distributed with representation (α, A) of order n. Then it follows that, for $a = -Ae$,

$$\begin{aligned} \hat{A}[H \otimes I_m + U] &= \int_0^\infty \exp\{(H \otimes I_m + U)t\}dA(t) \\ &= (\alpha \otimes I_n)(-(A \oplus (H \otimes I_m + U)))^{-1}(a \otimes I_n). \end{aligned} \tag{3.7}$$

Furthermore, the matrix η satisfies

$$\eta = (I_l \otimes \beta)(\alpha \otimes I_n)(-(A \oplus (H \otimes I_m + U)))^{-1}(a \otimes I_n)$$

and the following sequence of approximations converges to η: $\eta_0 = 0$,

$$\eta_{k+1} = (I_l \otimes \beta)(\alpha \otimes I_n)(-(A \oplus (H \oplus B + (I_l \otimes b)\eta_k))^{-1}(a \otimes I_n). \tag{3.8}$$

Proof: (**??**) follows directly from Asmussen (2000), p. 221, Proposition 1.7. The implicit equation for η results from substitution of (**??**) into (**??**). The convergence of the sequence $\{\eta_k\}$ follows directly from the proof to the lemma above, by observing that $(I_l \otimes b)\eta_k = (I_l \otimes b\beta)G_k$.

4. Integral Equation for η in the case of Phase-type Inter-claim Times

An alternative approach is to compute η directly in the case of phase-type inter-claim times using the methods of da Silva Soares & Latouche (2002). We assume the following phase type representations: claim amounts have representation (β, B) of order m, the horizon has representation (ν, H) of order l, and inter-claim times have representation (α, A) of order n.

Let us condition on an inter-claim time terminating during the interval $(y, y + dy)$, and consider what must happen during it: a) the likelihoods of the various states at the start of the interval are contained in the $l \times ln$ matrix $I_l \otimes \alpha$; b) the $ln \times ln$ matrix $\exp((H \oplus A)y)$ describes the evolution of the inter-claim time and the horizon during $(0, y)$; c) the claim occurs during $(y, y+dy)$ with probabilities recorded in the $ln \times l$ matrix $(I_l \otimes a)\ dy$. At this point, the horizon is "frozen" and its phase is retained. [The appropriate construct is the $l \times lm$ matrix $(I_l \otimes \beta)$.] Finally, d) the generator $U = I_l \otimes B + (I_l \otimes b)\eta$ must accumulate y revenue to set a new record level; the probabilities of the underlying states at the instant when this record is set are contained in the $lm \times lm$ matrix $\exp(Uy)$.

Removing the conditioning on y, and integrating from 0 to infinity yields

$$\eta = (I_l \otimes \alpha) \int_0^\infty \exp((H \oplus A)y)(I_l \otimes a)(I_l \otimes \beta) \exp(Uy)dy \quad (4.1)$$

$$= (I_l \otimes \alpha) \int_0^\infty \exp((H \oplus A)y)(I_l \otimes a\beta) \exp(Uy)dy. \quad (4.2)$$

since $(I_l \otimes a)(I_l \otimes \beta) = (I_l \otimes a\beta)$. The expression under the integral sign is similar to equation (12) in da Silva Soares & Latouche (2002) pertaining to fluid queues:

$$\Psi = \int_0^\infty \exp(T_{11}y)T_{12} \exp(Uy)dy. \quad (4.3)$$

There, T_{11} describes transitions among transient states in which fluid is continually increasing, and T_{12} describes transitions in which one moves from an increasing state to states in which fluid is diminishing. Equation (**??**) provides the probabilities, for all increasing states, that one will eventually return to the present fluid level, and be in the various decreasing states at the instant this occurs. Setting $T_{11} = H \oplus A, T_{12} = I_l \otimes (a\beta)$, one obtains the Ψ matrix via uniformization as in da Silva Soares & Latouche (2002). Then one evaluates $\eta = (I_l \otimes \alpha)\Psi$ and $U = I_l \otimes B + (I_l \otimes b\alpha)\Psi$.

The interested reader is directed to da Silva Soares & Latouche (2002) for a full discussion of the algorithm for the determination of Ψ and the theory behind it. Essentially, the approach employs an embedded quasi-birth-and-death (QBD) process, whose G matrix of first-passage probabilities contains the Ψ matrix in the upper right block.

Among the biggest advantages of this approach is the fact that the order of the matrices being determined is $l \times (m + n)$, which is typically much smaller than those needed using the matrix moment generating function, where the matrices are of size $l \times m \times n$.

5. The Stationary Sparre-Andersen Case

Typically claim occurrence processes are assumed to operate independently of the period when coverage comes into effect. The "stationary Sparre-Andersen" model reflects this by treating the time until the first claim occurs as a forward recurrence time. The only impact this introduces is a new probability matrix $\eta^{(s)}$ at the first ladder height. Denoting the distribution function of the forward recurrence time by $A^*(t)$, one obtains

$$\eta^{(s)} = (I_l \otimes \beta) \int_0^\infty \exp\{(H \otimes I_m + U)t\} dA^*(t) \tag{5.1}$$

$$\begin{aligned} &= (I_l \otimes \beta) \int_0^\infty \exp\{(H \otimes I_m + U)t\}(1 - A(t))dt/\mu_A \\ &= (I_l \otimes \beta)/\mu_A[\int_0^\infty \exp\{(H \otimes I_m + U)t\}dt \\ &\quad - \int_0^\infty \exp\{(H \otimes I_m + U)t\}A(t)dt]. \end{aligned} \tag{5.2}$$

The latter of these integrals can be simplified further, by relating it to the equation for η. Upon integrating (**??**) by parts, one finds

$$\eta = -(I_l \otimes \beta) \int_0^\infty \exp\{(H \otimes I_m + U)t\}A(t)dt \ (H \otimes I_m + U). \tag{5.3}$$

Substitution of this in the foregoing expression, and evaluation of the first integral yields

$$\eta^{(s)} = (\eta - I_l \otimes \beta)(H \otimes I_m + U)^{-1}/\mu_A. \tag{5.4}$$

Remark: No particular simplification of $\eta^{(s)}$ occurs in the Sparre-Andersen case. However, it does simplify to the form in Theorem 4.4, pp. 230-231 of Asmussen (2000) when there is no horizon. This is readily verified after tedious manipulations by selecting an exponential horizon in (**??**) and letting the rate approach 0.

6. Deficit at Ruin Prior to a PH Horizon

When ruin occurs prior to a phase-type horizon being reached, the size of the deficit at ruin is also phase-type. The probability vector $\nu\eta e^{Uu}$ contains the probabilities of being in the various pairings of horizon stage and claim phase at the moment of ruin. Aggregating appropriately over all possible horizon stages, we can obtain the proper initial probability vector for this phase type formulation, as the entire deficit comes from the claim causing ruin.

Corollary 5: The deficit at the time of ruin, given that ruin occurs prior to a phase-type horizon $\mathrm{PH}(\nu, \mathrm{H})$, has a phase-type distribution with representation $\mathrm{PH}(C\nu\eta e^{Uu}({e_l}^T \otimes I_m), B)$, where $C = [\nu\eta e^{Uu}({e_{lm}}^T)]^{-1}$.

Proof: Post-multiplication by the matrix $({e_l}^T \otimes I_m)$ performs the aggregation over all horizon stages, so that $\nu\eta e^{Uu}({e_l}^T \otimes I_m)$ is the distribution of the claim states at the instant of ruin. Furthermore, B is the intensity matrix of the phase changes of the payment interval. A similar line of thought, applied to the stationary renewal risk model, yields the following:

Corollary 6: The deficit at the time of ruin, given that ruin occurs prior to a phase-type horizon $\mathrm{PH}(\nu, \mathrm{H})$ in the *stationary renewal risk model* has a phase-type distribution with representation $\mathrm{PH}(C^{(s)}\nu\eta^{(s)}e^{Uu}({e_l}^T \otimes I_m), B)$, where $C^{(s)} = [\nu\eta^{(s)}e^{Uu}({e_{lm}}^T)]^{-1}$.

By choosing an Erlang-l distributed horizon and letting $l \to \infty$, weobtain asymptotically exact results for the deficit, when ruin occurs by time t in both models. This problem will be addressed in future work.

Remark: Recent other results establishing a phase-type form for the size of the deficit upon ruin include Drekic et al (2004) and Willmot et al (2004), both of which consider the Sparre-Andersen model with phase-type claim sizes. A flexible model allowing for correlation in the claim time and claim size processes also yields a phase-type form for the deficit upon ruin (see Badescu et al (2004, Corollary 4.5)), generalizing the foregoing results.

7. Numerical Examples

Two examples have been selected, to illustrate the accuracy and the speed of convergence of the approximation. The first is from Thorin & Wikstad (1973), where an advanced root-finding approach was used to determine finite time ruin probabilities. Their Table 8 provides finite time ruin probabilities for the case where the inter-claim time is a mixture of two exponentials, with an overall mean inter-claim time of 1. Their claim size distribution is a mixture of five exponentials used to *approximate* the Pareto distribution $F(y) = 1 - (1 + 2y)^{-3/2}$; see Thorin & Wikstad (1973) for the parameter values. We present in Table 1 below the results for the case where $c = 1$. (The other cases not presented here can be easily handled by an appropriate rescaling of the rates for the phase type inter-claim revenue.)

What Table 1 reveals is a rather quick approach towards the exact probability of ruin for small values of the Erlang order l. Frequently, the approximating probability of ruin when $l = 9$ is already within 1%, although the error is as large as 3% for the case where $T = 1000$ and for an initial surplus of $u = 100$. Clearly, even approximations of relatively small order yield results accurate enough to employ for any decision making purpose.

Table 1: Comparison of Approximate Ruin Probabilities with Thorin & Wikstad Results

T	u	T & W	$l = 1$	$l = 3$	$l = 5$	$l = 7$	$l = 9$
100	0	0.87387	0.83288	0.86316	0.86775	0.86959	0.87059
	100	0.04060	0.04024	0.04065	0.04066	0.04066	0.04065
	1000	0.00114	0.00114	0.00114	0.00114	0.00114	0.00114
	10000	0.00001	0.00001	0.00001	0.00001	0.00001	0.00001
1000	0	0.94074	0.92063	0.93585	0.93798	0.93883	0.93928
	100	0.29719	0.23975	0.27390	0.28263	0.28661	0.28888
	1000	0.01176	0.01212	0.01190	0.01184	0.01182	0.01181
	10000	0.00013	0.00014	0.00014	0.00014	0.00014	0.00014
10000	0	0.97261	0.96307	0.97038	0.97137	0.97177	0.97197
	100	0.63980	0.55334	0.61359	0.62473	0.62926	0.63171
	1000	0.13005	0.11556	0.12430	0.12643	0.12741	0.12797
	10000	0.00177	0.00212	0.00189	0.00184	0.00182	0.00181

Unfortunately, further gains in accuracy require much larger values of l. This is not surprising: as l increases, there are successively smaller differences in the Erlang distributions involved. Thus, if one wanted five digits of accuracy, one could anticipate a very large value of l and a long time to calculate. Luckily, much better accuracy (for effectively the same computational effort) is obtained when one makes use of the Richardson extrapolation employed in Asmussen et al (2002):

$$\psi_1(u, H_l) = (l + 1)\psi(u, H_{l+1}) - l\psi(u, H_l).$$

When this extrapolation is applied to the previous example, one obtains the approximate probabilities of ruin in Table 2. One sees immediately that a much better degree of accuracy is obtained, for lesser values of l. For instance, in the "worst case scenario" of Table 1 where $T = 1000$ and $u = 100$, the original approximate probability of ruin for $l = 9$ had a 2.8% relative error. A similar accuracy is obtained using the Richardson extrapolation with $l = 1$. The conclusion to be drawn is that by combining the Erlangian approximations with the Richardson extrapolation, accurate approximate values can be obtained quite readily.

Of the two methods presented in sections 6 and 7, the η recursion, which is based on inverting matrices of size $l \times m \times n$, was such that we were able to use Mathematica without meaningful delay, whenever l was less than 7. (By this we mean that results came back in a matter of seconds; they were available before the user was ready to proceed.) Even for $l = 9$, the algorithm produced results in less than a minute. In contrast, the second algorithm entailing the determination of the Ψ matrix as per da Silva Soares and Latouche (2002) works with matrices of size $l \times (m + n)$. Here results were obtained with no meaningful delay for values of l approaching 30.

Our second example was first proposed by Asmussen & Højgaard (1999), who presented a corrected diffusion approximation to approximate finite-time ruin probabilities in the Sparre-Andersen model. The claim-size distribution in their Example 4.1 approximates a lognormal distribution by a phase-type approximation of order four, and the interclaim times follow a distribution that is a mixture of four exponentials. Asmussen & Højgaard (1999) considered that no "exact" solution existed for the Sparre-Andersen case, yet our approach provides a sequence that is asymptotically exact. Furthermore, our computations involve straightforward evaluations, relative to the complicated determination of the β_1 coefficient in the corrected diffusion approximation (as observed by the authors).

Table 2: Comparison of Extrapolated Ruin Probabilities with Thorin & Wikstad Results

T	u	T & W	$l = 1$	$l = 3$	$l = 5$	$l = 7$
100	0	0.87387	0.88074	0.87483	0.87426	0.87408
	100	0.04060	0.04067	0.04065	0.04065	0.04063
	1000	0.00114	0.00114	0.00114	0.00114	0.00114
	10000	0.00001	0.00001	0.00001	0.00001	0.00001
1000	0	0.94074	0.94503	0.94128	0.94096	0.94088
	100	0.29719	0.28830	0.29534	0.29645	0.29680
	1000	0.01176	0.01180	0.01177	0.01176	0.01176
	10000	0.00013	0.00013	0.00013	0.00013	0.00013
10000	0	0.97261	0.97476	0.97290	0.97273	0.97269
	100	0.63980	0.64364	0.64160	0.64056	0.64018
	1000	0.13005	0.12786	0.12954	0.12979	0.12987
	10000	0.00177	0.00183	0.00178	0.00177	0.00177

We note that the $l = 7$ and extrapolated values presented in our table below are consistent with the trajectories of both the corrected diffusion approximation and the simulation, as presented in Figure 2 of Asmussen & Højgaard (1999). In this example, however, the extrapolated values for $l = 1$ are smaller than the regular values for $l = 7$ when $T = 10$, which is atypical. Furthermore, the extrapolated value for $l = 6$ when $T = 50$ actually exceeds the ultimate ruin probability very slightly. Nonetheless, we would submit that the accuracy of the results for $l = 7$ is already quite sufficient for any decision making purpose.

The numbers for this second example were run completely using the more time-consuming Mathematica code, yet still each value for $l = 7$ ran in about 30 seconds.

Table 3: Approximate Ruin Probabilities for Asmussen & Højgaard Example 4.1

T	$l = 1$	$l = 2$	Extrapolation	$l = 6$	$l = 7$	Extrapolation
10	0.01253	0.01424	0.01595	0.01590	0.01605	0.01695
20	0.01672	0.01909	0.02146	0.02113	0.02129	0.02225
30	0.01868	0.02107	0.02346	0.02283	0.02295	0.02367
40	0.01982	0.02207	0.02432	0.02349	0.02357	0.02405
50	0.02057	0.02265	0.02473	0.02379	0.02384	0.02414
1000	0.02391	0.02411	0.02431	0.02411	0.02411	0.02411

8. Acknowledgements

The work of Dr. Stanford has been supported by a "Research in Brussels" grant from the Gouvernement de la Région de Bruxelles – Capitale, as well as his NSERC Discovery grant, which has also supported the doctoral work of Dr. Badescu. The work of Dr. Breuer has been supported by the "Prix Ouverture International" of the Université Libre de Bruxelles. We also wish to thank the referees for their comments, which have improved the paper.

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