Some calculations and conjectures on a differential game motivated by the risk sensitive control of stochastic networks in the large deviations regime

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Abstract

We present below some differential games calculations/conjectures motivated by the "asymptotic risk sensitive" optimal control proposed by Whittle and Fleming, and recently implemented for queueing networks by Atar, Dupuis and Schwartz. More specifically, we derive optimal scheduling rules for a certain network structure, we explore a method of calculating the "constant asymptotic exponents" for the stationary distribution of networks, and we compute explicitly the optimal switching curve for a specific tandem example.

1 Introduction

1. **Motivation.** The field of stochastic networks is presently undergoing a period of intensive research, motivated by the need to control and improve the performance of complicated modern communication networks like the Internet, or of manufacturing systems, like for example of semiconductor chips. Often, these systems are modelled as multidimensional Markovian or renewal processes $\boldsymbol{X}(t)$ subject to boundary constraints, and their control ends up being formulated as a stochastic control problem of choosing time varying control parameters $\boldsymbol{u}(s)$, with the goal of optimizing costs of the form:

$$V(\boldsymbol{x}) = \inf_{\boldsymbol{u}} \mathbb{E}_{\boldsymbol{x}} \int_0^T h(\boldsymbol{X}(s))ds + G(\boldsymbol{X}(T)) \tag{1}$$

The horizon of interest is some stopping time T, $\boldsymbol{x} = \boldsymbol{x}(0)$ denotes the initial load, $\mathbb{E}_{\boldsymbol{x}}$ denotes expectation with respect to the stochastic model assumed, and $h(\boldsymbol{x}), G(\boldsymbol{x})$ are appropriately chosen holding/terminal costs reflecting the controller's objectives. For example, one might consider $h(\boldsymbol{x}(s)) = a\sum_i x_i^2(s)+$

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$\sum_i \kappa_i x_i(s) - c$ (with c huge, so that the controller attempts to maximize the working horizon, while avoiding huge inventory amounts x_i). With a Markovian model, the objective (1) will translate then (at least in a viscosity sense) into a HJB equation for the objective $V(\boldsymbol{x})$ of the form:

$$\inf_{\boldsymbol{u}} \mathcal{G}_{\boldsymbol{u}}\, V(\boldsymbol{x}) + h(\boldsymbol{x}) = 0, \qquad V(\boldsymbol{x}) = G(\boldsymbol{x}) \qquad \text{if } x \in \partial \tag{2}$$

where $\mathcal{G}_{\boldsymbol{u}}$ denotes the operator associated with the Markovian model considered (which further depends on parameters $\boldsymbol{u}$ at the disposition of the controller).

In one dimensional setups, (2) admits often closed form solutions (see for example [1], [14]). In several dimensions however, excepting some notable exceptions like the multi-armed bandit problem, the solution of(2) is only feasible numerically and gets quite challenging as the number of dimensions increases. Therefore, researchers were naturally lead to approach the solution of HJB equations associated to network control problems via asymptotic approximations, obtained via a mean-field/fluid, a diffusion, or a large deviations limit.

2. **Asymptotic approaches.**

- The **fluid limit** reflects the behavior of systems viewed from far away, in a limit under which "the stochasticity vanishes altogether" and the systems evolve deterministically along their expected mean field lines.
- The diffusion limit is a second order refinement of the fluid limit.
- The large deviations limit reflects the rare behavior of systems conditioned on some unusual behavior having been observed, like for example on the escape to some far away boundary, lying in a direction opposite to the mean field paths.

The fluid limit ends up replacing the nonlinear HJB equation (2) by the **first order** HJB equation:

$$\inf_{\boldsymbol{u}} \mathcal{A}_{\boldsymbol{u}}\, V(\boldsymbol{x}) + h(\boldsymbol{x}) = 0, \qquad V(\boldsymbol{x}) = G(\boldsymbol{x}) \qquad \text{if } x \in \partial \tag{3}$$

where $\mathcal{A}_{\boldsymbol{u}}$ is the first order operator $\boldsymbol{f}_{\boldsymbol{u}} \cdot \nabla$ and $\boldsymbol{f}_{\boldsymbol{u}}$ denotes the mean-vector field, i.e. the derivative with respect to time of the expectation $\boldsymbol{x}(t) = \mathbb{E}X(t)$ of the (unconstrained) network process $X(t)$; there is also an equivalent optimal control formulation, given in (12) below.

While very successful qualitatively [5], [12], [24], [60], the fluid approximation suffers from the drawback of ignoring all information on the original model beyond its means. On the other hand, the diffusion approximation, despite a reduction in the number of parameters, still leaves one obliged to resort to numerical methods for solving nonlinear (second order) multidimensional PDE's.

3. **Risk sensitive optimization.** A third alternative is to adopt the so called "risk sensitive" exponential objective proposed by Fleming and Whittle:

$$V(\boldsymbol{x}) \quad = \quad \inf_{\boldsymbol{u}} \mathbb{E}_{\boldsymbol{x}} e^{\int_0^T h(\boldsymbol{X}(s))ds + G(\boldsymbol{X}(T))} \tag{4}$$

under which the value function will satisfy (at least in a viscosity sense) the nonlinear HJB:

$$\inf_{\boldsymbol{u}} \mathcal{G}_{\boldsymbol{u}} V(\boldsymbol{x}) + h(\boldsymbol{x})V(\boldsymbol{x}) = 0, \qquad V(\boldsymbol{x}) = e^{G(\boldsymbol{x})} \text{ if } x \in \partial \tag{5}$$

Besides weighing more (or less) heavily exceptionally large values of the functional $\int_0^T h(\boldsymbol{X}(s))ds + G(\boldsymbol{X}(T))$ and thus expressing attitudes towards risk, this objective has also the advantage of becoming tractable analytically under certain limits, just as fluid control, while retaining information on the stochastic model assumed under the form of the "large deviations" functional. More precisely, asymptotically one arrives at the first order Isaacs equation:

$$\inf_{\boldsymbol{u}} \sup_{\bar{\boldsymbol{r}}} \mathcal{A}_{\{\boldsymbol{u},\bar{\boldsymbol{r}}\}} V(\boldsymbol{x}) + h(\boldsymbol{x}) - L(\boldsymbol{x}, \bar{\boldsymbol{r}}, \boldsymbol{u}) = 0, \qquad V(\boldsymbol{x}) = G(\boldsymbol{x}) \qquad \text{if } x \in \partial$$

where $\mathcal{A}_{\{\boldsymbol{u},\bar{\boldsymbol{r}}\}}$ denotes a modified mean-field operator with new parameters $\bar{\boldsymbol{r}}$, and $L(\boldsymbol{x}, \bar{\boldsymbol{r}}, \boldsymbol{u})$ denotes a "large deviations/free energy" functional –see below for more details. The large deviations functional may be interpreted as a payoff to be paid by an opponent for changing the original "parameters" $\boldsymbol{r}$ of the process to new parameters $\bar{\boldsymbol{r}}$. There is also an equivalent differential games formulation, to be presented in (13) below. The exact meaning of "large deviations" functional and of "parameters" will depend on the specific model.

In conclusion, we have at our disposal two related asymptotic approaches – the fluid/mean-field limit and the large deviations limit (which are used typically in connection with the objectives (1) and (4) respectively), which share the common property of converting stochastic control problems into deterministic control/differential games problems. These two limits reduce the original nonlinear HJB problems to first order problems which may be attacked via the method of **characteristics**. This constructs explicitly the optimal control paths, priority domains and switching curves, providing thus valuable insights into the original stochastic control problems.

The validity of the fluid limit approach for queueing networks control has been thoroughly explored, for example by Meyn [46] and Maglaras [40]. On the other hand, the risk sensitive/large deviations control has been much less explored. Up to now, the convergence of an original stochastic control problem to a differential game has only been established in a few cases, like Fleming and Zhang [28] and Atar, Dupuis and Schwartz [3].

4. **Previous asymptotic control literature**. The fluid and large deviations asymptotic approaches, analog to the classical mechanics approximation of

quantum physics, have become quite popular among probabilists following the works of Kolmogorov, Kac, Wentzell, Freidlin, Fleming, and Varadhan. In particular, the use of the fluid approximation for network control was advocated from the start by G. Newell [47], L. Kleinrock [34], Segal and Moss [53], [54] and Filipiak [26]. However, in the following decade, while considerable progress was made into the understanding of Brownian networks and of loss networks, the potential of deterministic control (which is not so apparent in these contexts) was lost out of sight. Interest in fluid control started picking up again in the nineties, following the realization of the fact that fluid models capture the stability properties of stochastic networks [51], [39], [22], [19]. At the same time, Chen and Yao [18] provided a simple algorithm for greedy (not necessarily optimal) fluid control, Avram, Bertsimas and Ricard [5] showed that the optimal sequencing fluid control problem is tractable analytically, and Kumar and Perkins [49] developped a quadratic programming approach for the particular case of tandem lines. The key to tractability in the last two papers was the (independent) adoption of a transient linear objective of minimizing:

$$V(\boldsymbol{x}) = \int_0^T \sum_{i=1}^{I} c_i x_i(s) ds$$

where $\boldsymbol{x} = \boldsymbol{x}(0)$ denotes the initial load of the network and T is its **emptying** time. The linear objective enabled [5] to calculate piecewise linear dynamic priority indices (related to the "costate variables/Pontryaghin multipliers" of optimal control) and to obtain the first explicit expressions for optimal switching curves in network control. This yielded intricate optimal policies for scheduling multiclass networks [5], for routing [8], and for the optimal scheduling of fluid reentrant lines [6], [57], [58], [59]. Powerful numerical algorithms were developped by Bertsimas and Luo [12] and by G. Weiss [60], who used the duality approach for continuous linear programming of Pullan [50]. However, despite the considerable insights it yields, fluid control suffers from the major drawback of ignoring all information beyond the means.

5. **Pursuit evasion network differential games.** As pointed out by Whittle [61] and Fleming (see for example [27]), optimal control with an exponential/risk sensitive optimization objective leads asymptotically to pursuit evasion differential game problems, who share some of the tractability of fluid control problems, while attempting nevertheless to use more sophisticated cumulant information beyond the means. These differential games generalize fluid control in the sense that there appears now a second controller ("nature") with an objective opposed to the first controller, which is allowed to change the rates of the processes at the cost of having to pay a "large deviations/free energy" functional. The first attempt to use pursuit evasion games for the control of networks was made in Avram [8] (for a multidimensional reflected Brownian motion problem). The analytical tractability of this approach was demonstrated in several papers of Ball & al and Day – see for example [11], [20] – who worked in a framework corresponding to a deterministic fluid model

for service and Gaussian uncertainty for inputs. The first rigorous derivation of a pursuit evasion differential game as a limit (under an appropriate scaling of rates and costs) for a stochastic control problem seems to be Fleming and Zhang [28]. Recently, a further strong rationale for using this "robust control" approach came from the work of Atar, Dupuis and Schwartz [3], [4] on the classical model of Markovian queueing networks with state space Z_+^I. These authors showed that a pursuit evasion type differential game emerges also under a more natural large space limit, for the risk sensitive control problem of reaching as late as possible domains which do not include the attractive point of the fluid flow (hence the name "control in the large deviations regime").

To explore the computational potential of this approach, we consider below some pursuit evasion games which are close to, but do not quite fit in the framework of the result of [3]. We allow more general network structures, general intertransition times and arbitrary convex non-decreasing holding costs. All these variations matter little for the variational computations and help focus on one of the main advantages of the differential games asymptotic approach, the possibility of computing the "bicharacteristics" (i.e. the saddle-curves along which the controller and "nature" bring the system to catastrophe). We chose our specific differential game, to be called in the sequel a **large deviations network** (LDN) game with the goal of unifying the analytic manipulations necessary in several related network control problems, and also since the (LDN) game is close enough to the Atar/Dupuis/Schwartz game to warrant the conjecture that it will indeed provide the asymptotics of the corresponding risk sensitive stochastic control problem in the large deviations regime. We note that some aspects of this natural conjecture are currently being explored in a forthcoming paper of Schwartz and Weiss. For others however, the justification of the formal calculations might be a long way ahead.It seems therefore that the publication of some formal calculations and conjectures ahead of their rigorous justification might be profitable in this domain, with the obvious caveat.

For the potential advantages, it is worth recalling that the asymptotic approach yields insights some of the most important exact analytical results available for queueing networks. For example, recall the famous Jackson result that "Jackson networks" on $\mathbb{Z}_+^I$ have stationary distributions with an explicit product form, which is clearly a very particular case in view of the fact that a product form $\prod_{i=1}^I \rho_i^{n_i}$ determined by I parameters $(\rho_i, i = 1, ..., I)$ must end up satisfying simultaneously 2^I equilibrium equations. However, as shown in [10] (following works of Knessl and Tier [35], [36], Ignatyuk, Malyshev and Scherbakov [31], Ignatyouk [32], Avram, Dai and Hassenbein [7], [9], and Fujimoto, Makimoto and Takahashi [29]), this result has a natural extension to the stationary distribution of queueing networks with renewal intertransition times (as well as to reflected Brownian motion networks). Namely, these stationary distributions admit "locally dominant product form approximatons", whose exponents may be found by solving algebraically certain systems of

local equilibrium systems. The exact method of generating those local systems became however apparent only after examining the corresponding large deviations variational problem– see [32], [10] and below.

6. **Contributions and contents**. This paper attempts to review several developments in the "budding" new domain of large deviations network games, to be defined in section 2. In section 3 we provide general sequencing and routing rules for a certain network structure (which extend corresponding fluid control rules of [5]). In section 4 we find explicitly the switching curve for a differential game corresponding to the optimal sequencing of a tandem, allowing for Erlang E_n transition times (which allows recovering the corresponding fluid result of [5] in the limit $n \to \infty$).

2 A stochastic model and an associated large deviations network game

We describe now the stochastic model which suggested the asymptotic variational formulation resolved here: that of a "renewal queueing" process $\tilde{\boldsymbol{X}}(t)$ on some state space $S \subset \mathbb{Z}_+^I$, which models the number of customers waiting for service in I queues under various possible constraints. In the absence of constraints, our process, to be called "free renewal process" $\boldsymbol{X}(t)$, will evolve due to a set K of independent pure jump processes ("activities"), by which we mean processes of sums of fixed direction integer jumps, which occur each after independent interarrival times, at average rates r_k (=reciprocals of expected inter transition times).

1. **The "free" renewal queueing network process.**

   Definition 1 *A "free renewal" queueing network process $\boldsymbol{X}(t)$ is a pure jump process $\boldsymbol{X}(t) = \sum_{k \in K} \boldsymbol{d}_k N_k(t)$ on $\mathbb{Z}^I$, where $\boldsymbol{d}_k, k = 1, ..., K$ represent fixed jump directions on $\mathbb{Z}^I$ for the transitions occurring at the increase times of a set of counting processes $N_k(t)$, assumed* **independent***, with stationary independent increments and cumulant generating functions*

   $$\kappa_k(s) = \lim_{t \to \infty} \frac{\log E e^{s\, N_k(t)}}{t}$$

   assumed to exist for some strictly positive number s.

   Notes:

   (a) The independence of the activities ensures that the cumulant generating function of the free process $\boldsymbol{X}(t)$,

   $$C(\boldsymbol{\alpha}) = \lim_{t \to \infty} \frac{\log E e^{\boldsymbol{\alpha}\, \boldsymbol{X}(t)}}{t}$$

satisfies

$$C(\boldsymbol{\alpha}) = \sum_{k \in T} \kappa_k(s_k) \tag{6}$$

where $s_k = \boldsymbol{\alpha}\, \boldsymbol{d}_k$, for all values $\boldsymbol{\alpha}$ for which it is well defined. This expression, to be called the **fundamental form** of the network, **is sufficient for deriving formally all the "network exponents"**.

(b) We will adopt the convention that $\boldsymbol{\alpha}$ denotes a row vector and $\boldsymbol{X}(t), \boldsymbol{d}_k$ denote column vectors.

(c) In the renewal case, the asymptotic cumulant generating function of a stationary renewal counting process $N(t)$ is provided by the classical Cramer-Lundberg equation, in terms of the Laplace transform $\hat{f}$ of the interrarrival density $f(x)$.

Proposition: (Cramer-Lundberg) For $s > 0$, the asymptotic cumulant generating function $\kappa(s) = lim_{t \to \infty} \frac{logEe^{s\ N_t}}{t}$ of a renewal counting process $N(t)$ is the unique root of the equation

$$\hat{f}(\kappa) = e^{-s} \tag{7}$$

(d) An important role below will be played by the convex duals of the cumulant generating functions $\kappa_k(s)$, defined by $L_k(\bar{\mu}) = \max_s s\, \bar{\mu} - \kappa_k(s)$.

i. For exponential interransition times distributions with mean μ_k, the cumulant generating functions are given explicitly by $\kappa_k(s) = \mu_k(e^s - 1)$, and the convex duals are $L_k(\bar{\mu}) = \mu_k l(\frac{\bar{\mu}}{\mu_k})$ where

$$l(x) = x\ log(x) - x + 1, \tag{8}$$

is the large deviations functional associated to a Poisson process of rate 1.

ii. For Erlang interarrival-times with expectation μ_k and n_k stages, the cumulant generating functions are $\kappa_k(s) = n_k \mu_k(e^{s/n_k} - 1)$ and the associated "convex duals" are $L_k(\bar{\mu}) = n_k \mu_k l(\frac{\bar{\mu}}{\mu_k})$.
The class of Erlangian networks includes for example that of Markovian networks (corresponding to $n_k = 1, \forall k$), that of fluid networks (corresponding to $n_k = \infty, \forall k$), as well as mixtures of the two.

(e) Let A denote the $I \times K$ "structure matrix" whose columns are the directions $\boldsymbol{d}_k$. The expectation $\boldsymbol{x}(t) = \mathbb{E}\boldsymbol{X}(t)$ satisfies the differential equation

$$\dot{\boldsymbol{x}}(t) = A\boldsymbol{r}$$

where $\boldsymbol{r} = (r_k, k \in K)$ with $r_k = \kappa_k'(0)$ are the rates of the activities. This is precisely the asymptotic "mean-field/fluid dynamics" [17].

(f) A "free renewal queueing process" is entirely characterized by the set $\boldsymbol{\kappa} = (\kappa_k(s), k \in K)$ and by the structure matrix A, or, alternatively, by the cumulant generating function $C(\boldsymbol{\alpha})$.

Examples:

(a) **Markovian networks** are renewal network processes $X(t) = \sum_{k=1}^{K} \boldsymbol{d}_k N_k(t)$ for which the inter transition times of all the renewal processes $N_k(t)$ are exponentially distributed; equivalently, the cumulant generating functions are given explicitly by

$$\kappa_k(s) = r_k(e^s - 1)$$

(b) **Jackson networks** are skip free Markovian networks on $\mathbb{Z}_+^I$ for which only single arrivals, single departures and transfers are allowed. The respective rates are traditionally denoted by $\lambda_i, \mu_i p_{i,0}, \mu_i p_{i,j}$, where $p_{i,0} = 1 - \sum_{j=1}^{I} p_{i,j}$. Thus, μ_i represents the sum of all the rates which decrease the i' th coordinate, $P = \{p_{i,j}\}_{i,j=1}^{I}$ is the matrix giving the proportions of inner routings and $p_{i,0}$ give the proportions who leave the system.

(c) **Erlangian networks**. We will be working below at most with networks with Erlangian intertransition times. A similar treatment of networks with general phase–type activities is possible, at the price of introducing more parameters.

2. **"Single departure" networks.** We assume that the classes of jobs are further partitioned into a set J of groups called "stations", which will reflect service constraints. We denote by $C(j)$ the set of classes served at a station $j \in J$, and by $S(i) \in J$ the station where class i is served. Note that to model a particular type of jobs flowing through a network, we need to define a different class modeling the flow at each station.

We will restrict ourselves to queueing networks with a particular combinatorial structure.

Definition 2 *A single departure network has all activities $k \in K$ defined as multiple transfers along a set of oriented* **multiedges**, *starting from* **at most one** *starting class $s(k)$ to a* **set** *of ending classes $e(k)$. With each edge there are also associated integer weights indicating the multiplicity of the transfer. Algebraically, each activity is an integer vector having at most one negative entry of exactly -1.*

Notes:

(a) This definition allows also for a set of "arrival activities" from the outside, denoted by K_a, which have no starting class, and whose rates will be denoted by λ_k.

(b) The structure matrix A of our model is precisely the incidence matrix of this directed graph on the set of all classes I (with additional "arrival multiedges" which have no starting point within I). Its column $\boldsymbol{d}_k$

corresponding to an activity k will be of the form:

$$A_{ik} = \begin{cases} -1 & \text{if } k \text{ depletes class } i, \text{ i.e. } i = s(k) \\ 1 & \text{if } k \text{ increases class } i, \text{ i.e. } i \in e(k) \\ 0 & \text{otherwise.} \end{cases}$$

(thus the columns of the arrival activities $k \in K_a$ will have the -1 entry of the source missing).

Definition 3 *If $\mathcal{N} = (A, \boldsymbol{\kappa})$ is a single departures* **Markovian** *queueing network, then for any row i the shorted network $\tilde{\mathcal{N}}_i$ is a network obtained as follows: 1) The class i is removed and all transitions not involving it are kept unchanged. 2) The transitions involving the i'th class are partitioned into the sets $\mathcal{A}_i$ and $\mathcal{D}_i$ of transitions which increase and respectively decrease the class i. These transitions are then replaced in the new network by the set of all sums $\boldsymbol{d}_{(k,k')} = \boldsymbol{d}_k + \boldsymbol{d}_{k'}$, where $\boldsymbol{d}_k \in \mathcal{A}_i$ and $\boldsymbol{d}_{k'} \in \mathcal{D}_i$. 3) New cumulant generating functions $\kappa_{(k,k')}$ are associated with these new activities, by choosing new exponential rates of $r_k \frac{r_{k'}}{\sum_{k' \in \mathcal{D}_i} r_{k'}}$.*

Notes:

(a) For the Jackson case, a "shorting" recipe appeared first in section 5 of [32].

(b) For the case of general renewal networks, shorting has been defined and used in [10] as follows:

Definition 4 *Let $R^{(i)}(\boldsymbol{\alpha}) = \sum_{s(k)=i} \kappa_k(s_k)$ denote the cumulant generating function of all the transitions in $\mathcal{D}_i$ (which become impossible when $x_i = 0$). Let $C_i(\boldsymbol{\alpha})$ denote the "***reduced cumulant generating function*** " obtained by eliminating in the global cumulant generating function $C(\boldsymbol{\alpha})$ the twist α_i, by using the equation $R^{(i)}(\boldsymbol{\alpha}) = 0$.*

Conjecture 1 *The reduced cumulant generating function $C_i(\alpha)$ is the cumulant generating function of a free renewal queueing network process.*

The question on how to define precisely this new network implicit in the conjecture, while very interesting, is outside the scope of our paper. We refer however to the "boundary network" processes obtained by projecting an original network on a boundary facet Λ, described in [25].

Note: The shorting procedure may give rise to activities involving vectors $\boldsymbol{d}_k$ with components larger then two, the shorting of which may not be carried out. There is no such problem for Jackson-type and feedforward networks. However, we prefered to single out the larger class of single departure networks where the procedure does not always work for all orders, since this raises the interesting question of whether local product approximations corresponding to those orders are still possible.

3. **The " polyhedrally regulated" network process**. We will consider queueing network processes $\tilde{\boldsymbol{X}}(t)$ which are obtained from a free queueing process by constraining it to remain in some polyhedral state space S, via the simplest boundary mechanism which consists simply in cancelling transitions which would take the process outside the state space. This mechanism, to be called **regulation**, is a particular case of Skorohod mapping.

   Definition 5 *A "polyhedrally regulated" queueing process $\tilde{\boldsymbol{X}}_t$ is the process obtained by cancelling all the activities of a free process $\boldsymbol{X}(t)$ which would take it outside a polyhedral state space S.*

   Notes:

   (a) The boundary regulation mechanism will be typically different on each boundary facet. For example, in the case of queues with unbounded buffers, the state space is the I dimensional quadrant $\mathbb{Z}_+^I$ and we might have up to $2^I - 1$ different boundary mechanisms. Additional mechanisms will appear at each boundary of the state space caused by limited buffer capacities or other constraints[1].

   (b) The regulation process for our model may be achieved by the controller shutting off partially activities which would take the process outside its state space.

   In conclusion, a **polyhedrally regulated renewal network** is a triple (A, S, L) formed of:

   (a) A linear operator (structure matrix) A yielding the possible jump directions (and determining the velocity $\boldsymbol{x}(t) = \mathbb{E}\boldsymbol{X}(t)$ of the expected uncontrolled free process in the interior of the state space via $\dot{\boldsymbol{x}}(t) = A\,\boldsymbol{r}(t)$).

   (b) A polyhedral state space S whithin which evolves the content of the network.

   (c) A convex "large deviations/free energy" functional $L(\boldsymbol{x}(t), \bar{\boldsymbol{r}}(t))$, where $\boldsymbol{r}$ denote informally "the parameters of the original model which change under an Kramer/exponential transform of the measure". We will consider mostly the case of separable large deviations functionals $L(\boldsymbol{x}, \bar{\boldsymbol{r}}) = \sum_{k \in K} L_k(\bar{r}_k)$ with L_k being the convex duals of the cumulant generating functions $\kappa_k(s)$; with exponential or Erlang transitions, $\boldsymbol{r}$ may be taken as the reciprocals of the interarrival times. We also consider briefly the case of a quadratic nonseparable L, arisings in reflected Brownian motion

[1] There is nothing fundamentally different about the state space $\mathbb{Z}_+^I$ to distinguish it from other polyhedral state spaces, beyond the convenience of having only one corner. For a different (maybe bounded) state space S, one should consider a sequence of state spaces of the form $S^{(n)} = nS \bigcap \mathbb{Z}_+^I$ chosen so that after scaling by n they reduce to the fixed polyhedron S. If S has more than one vertex, some of the results below would need to be applied separately at each vertex of the state space, in terms of the local variables defining it.

networks (when $|K| = I$), where $\boldsymbol{r}$ denotes the "shifted" drifts under a Kramer/exponential transform.

Note: The simplest and most general possible definition of the large deviations functional L is as the convex dual of the cumulant generating function of the free process $C(\boldsymbol{\alpha})$. This general approach stops one however from exploiting convenient features of a particular model like the independence of the activities. It seems profitable therefore to allow for "model dependent" definitions of L and $\boldsymbol{r}$, as we've done above, (with the knowledge that they will be related to the general definition via the "contraction principle"). For a recent example where it is not most natural to parametrize L by the "actual rates", see [38].

4. **Controlled networks with service constraints.** We consider now the effect of service constraints on our model.

   Each activity k will have a control variable u_k associated (possibly constrained to equal 1), specifying the fraction of time that the station processing the class $s(k)$ works on it, with the exception of the arrival activities.

   Assumption A: Arrival activities K_a are **uncontrolled**, i.e. $u_k = 1, \forall k \in K_a$.

   If all the effort of the station $S(s(k))$ would be devoted to an activity k, and the amount of the $i = s(k)$ class jobs would never become 0, then the completion times for this activity would form a renewal process, with cumulant generating function $\kappa_k(s)$. However, the service constraints will have as effect that certain activities will work only a fraction of time $u \in [0, 1]$, with the effect that the cumulant generating function will change to $u\ \kappa_k(s)$ and the large deviations functional will chage to $u\ L_k(\frac{\bar{r}}{u})$.

   Assumption B: The stations of the network are partitioned into three disjoint sets J_r, J_e, J_s of "**discretionary routing**", **class controlled**" and "**sequencing**" stations, with similar notation $I_r, I_e, I_s / K_r, K_e, K_s$ for the corresponding classes and activities. Groups of discretionary routing activities with the same starting point $s(k) = i, i \in I_r$ are assumed to have a common class dependent "stochasticity" $\kappa_i(s)$, while the other activities are assumed to have an activity dependent "stochasticity" $\kappa_k(s)$. The control variables are assumed to be activity dependent for the routing and sequencing activities. For the class controlled activities, we assume the control variables of activities k sharing the same starting point $s(k) = i, i \in I_e$ to be equal, with $u_k = \frac{u_i}{|\{k:s(k)=i\}|}$, where u_i denotes the total effort for class i. These three situations (which will give rise to different optimality rules) unify lots of previous work.

   The union of all controllable activities will be denoted by $K_c = K_r \cup K_e \cup K_s$ (and $K = K_a \cup K_c$ denotes the union of all activities). The controllable activities satisfy "station" constraints:

$$\sum_{\{k:s(k)\in C(j)\}} u_k \leq 1, \forall j \in J$$

which may be written in matrix form as:

$$D\,\boldsymbol{u}^{(c)}(t) \le \mathbf{1} \tag{9}$$

where D is a $|J| \times |K_c|$ "partition matrix" (meaning a matrix with exactly one 1 in each column, and zeroes else):

$$D_{jk} = \begin{cases} 1 & \text{if station } j \text{ performs the activity } k \\ 0 & \text{otherwise.} \end{cases}$$

Occasional additional equality constraints, like those necessary to describe the class controlled stations, will be denoted by

$$D_e \boldsymbol{u}^{(e)} \;=\; \boldsymbol{e} \tag{10}$$

for some subset of the control variables to be denoted by $\boldsymbol{u}^{(e)}$.

5. **Fluid networks control.** We suppose that holding numbers x_i of the serviced classes $i \in I$ at their respective station involves a unit time holding cost $h(\boldsymbol{x}) = h(x_1, x_2, ..., x_{|I|})$, and consider now the mean-field/fluid control problem of optimizing the integrated cost (1) with $G(\boldsymbol{x}) = 0$, for the **expectation** of our stochastic model (this coincides with the approximate asymptotic control emerging from a far away limit). Let $x_i(t)$ denote the expected number of jobs of class i in the system at time t. Let $\boldsymbol{r} = (\lambda_k, k \in K_a) \cup (\mu_k, k \in K_c)$ denote the vector of rates of all the activities and let $\tilde{\boldsymbol{r}}(t) = \boldsymbol{r}(t) \bullet \boldsymbol{u}(t)$ denote the vector obtained by the componentwise product of $\boldsymbol{r}(t), \boldsymbol{u}(t)$ (with the obvious interpretation of the controller partially/shutting off some of the activities). The evolution equation of the controlled network is:

$$\dot{\boldsymbol{x}}(t) = A\,\tilde{\boldsymbol{r}}(t) \tag{11}$$

where A denotes the $|I| \times (|K|)$ structure matrix expressing the relationship between classes and activities.

Let T denote the emptying time of the network (which will be finite under some control policy, provided that stability conditions hold). Combining the controlled fluid evolution (11), the constraints (9), (10) and a minimum total holding cost objective, we arrive at the fluid control problem:

$$\begin{aligned} (FLUID\ CONTROL)\ v(\boldsymbol{x}) &= \min \int_0^T h(\boldsymbol{x}(t))dt \\ \dot{\boldsymbol{x}}(t) &= A(\boldsymbol{r} \bullet \boldsymbol{u})(t) \\ \boldsymbol{x}(t) &\ge \mathbf{0} \\ D\boldsymbol{u}^{(c)}(t) &\le \mathbf{1} \\ D_e\boldsymbol{u}^{(e)}(t) &= \boldsymbol{e} \\ \boldsymbol{u}(t) &\ge \mathbf{0} \\ \boldsymbol{x}(0) &= \boldsymbol{x} \end{aligned} \tag{12}$$

Problem $(FLUID\ CONTROL)$ may be viewed as a linear program in infinite dimensions. The number of constraints is finite, but the number of variables is infinite, since we have an $\boldsymbol{x}(t)$ vector for each infinitesimal time period $[t, t+\Delta]$. Such problems can be solved numerically by using a discrete approximation and solving a linear program in a finite (albeit large) number of variables [60]. It is also a calculus of variations problem, and maybe solved therefore via the Pontryagin maximum principle; for example, with linear holding costs, it has been solved in [5] (see also [8]).

6. **Large deviations pursuit evasion network differential games.** Let $h(\boldsymbol{x}(t))$ be a convex, nondecreasing holding cost (typically constant, linear or quadratic), to be minimized until a certain abnormal working domain ∂ (not including the asymptotic fixed point of the fluid field) is reached. We will formulate now a differential game conjectured to yield the asymptotic behavior in the large deviations regime of corresponding stochastic control problems of minimizing expected exponential "holding costs"

$$V(\boldsymbol{x}) = \min_{\boldsymbol{u}} \; E_{\boldsymbol{x}} \; e^{\int_0^T h(\tilde{\boldsymbol{X}}(s), \boldsymbol{u}(s)) ds}$$

for renewal networks $\tilde{\boldsymbol{X}}(t)$, where $\boldsymbol{x} = (x_i(0), i = 1, ..., I)$ denote the sizes of the various classes initially, and T is the first "**catastrophe**" time when the set ∂ is entered. We denote:

- by $\boldsymbol{r}(t)$ the usual rates of the activities.
- by $\bar{\boldsymbol{r}}(t)$ the "catastrophe rates" and
- by $\tilde{\boldsymbol{r}}(t) = \bar{\boldsymbol{r}}(t) \bullet \boldsymbol{u}(t)$ the "controlled catastrophe rates".

Using an activity k only in proportion u results in replacing its large deviations functional L_k by $uL_k(\frac{\tilde{\boldsymbol{r}}_k}{u}) = uL_k(\bar{\boldsymbol{r}}_k)$; note the interesting probability interpretation of simply multiplying the large deviations functional of the uncontrolled activity by the proportion of time it works. In conclusion, the resulting large deviations network differential game is:

$$\begin{aligned}
(LDNG)\; v(\boldsymbol{x}) &= \min_{\boldsymbol{u}} \max_{\bar{\boldsymbol{r}}} \int_0^T \big(h(\boldsymbol{x}(t)) - \boldsymbol{L}(\boldsymbol{x}(t), \bar{\boldsymbol{r}}(t))\, \boldsymbol{u}(t)\big) dt \qquad (13)\\
\dot{\boldsymbol{x}}(t) &= A\, \tilde{\boldsymbol{r}}(t)\\
\boldsymbol{x}(t) &\in S\\
D\boldsymbol{u}^{(c)}(t) &\leq \mathbf{1}\\
D_e\boldsymbol{u}^{(e)}(t) &= \boldsymbol{e}\\
\boldsymbol{u}(t) &\geq \mathbf{0}\\
\boldsymbol{x}(0) &= \boldsymbol{x},\; x(T) \in \partial\\
A\, \bar{\boldsymbol{r}}(t) \text{ points} &\quad \text{outside } S \text{ while } \boldsymbol{x}(t) \in \text{ a facet of the boundary}
\end{aligned}$$

where the optimization is only over paths which attain the target in finite time.

Notes:

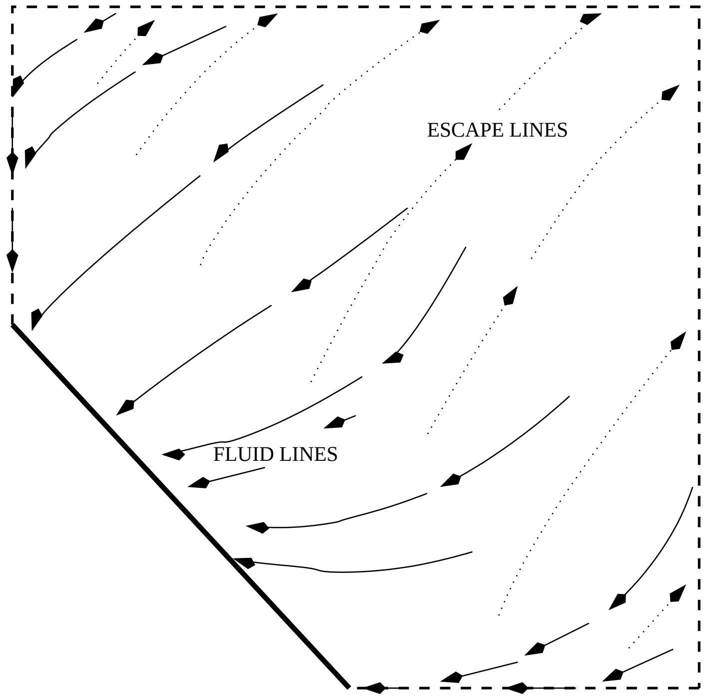

Figure 1: **Fluid/mean field and escape (conditioned) flow lines**

(a) We note first that when the target set $\partial = 0$ includes the attractive center for the fluid field, the (LDN) differential game (13) reduces to minimum cost fluid control, since the mean-field paths will become optimal and along them the entropy is 0. On the other hand, with the target being the complement of the rectangle $\partial = \{\boldsymbol{x} : x_i \leq z_i, \forall i\}$ and $h(\boldsymbol{x}) = -c$, with c a positive constant, (13) becomes the maximum exit time differential game considered recently in [3], [4]. Furthemore, with $c = 0$ we conjecture that it reduces to the large deviations variational problem for the stationary distributions of networks [2]. However, while fluid control and large deviations variational problems seemed to involve only the simplest from the "zoology of singular surfaces" of differential games, the differential game formulation involves more complicated members of this bestiary, like "dispersal surfaces" and "attractive surfaces", as may be gleaned from Ball and al [11] and Day [20].

(b) The heuristic interpretation of the game is that an opponent ("the stochasticity" or "nature") choses a new vector $\bar{\boldsymbol{r}}(t)$ of rates of transition (one for

[2]The large deviations variational problem problem has a very long history, starting with the one dimensional Cramer- Lundberg approximation of ruin probabilities. Among the vast contemporary literature, the most related works are Dupuis, Ellis and Weiss [21], Tsoucas [56], Ignatyouk, Malyshev and Scherbakov [31], Schwartz and Weiss [52], Borovkov and Mogulskii [15], [16], OConnell [48], Anantharam and Ganesh [30], Fujimoto, Makimoto and Takahashi [29], Ignatyouk [32], Atar and Dupuis [2] and Avram [10] for discrete networks, and Majewski [42], [43], [44], Avram, Dai and Hasenbein [3] and Dupuis and Ramanan [23] for reflected Brownian motion networks.

each activity) to replace the original ones r_k, which are the worse possible from the controller's point of view). However, the opponent must pay a price of $L(\boldsymbol{x}(t), \bar{\boldsymbol{r}}(t))$. Note that the "catastrophe" rates $\bar{\boldsymbol{r}}$ will depend on the optimization objective, unlike the fixed rates used in fluid control.

(c) The control parameters $\boldsymbol{u}$ play the double role of optimizing the network performance and of enforcing the constraint of keeping the process $\boldsymbol{x}(t)$ within S (by diminishing the transitions towards outside). Thus, the first constraint describes simultaneously the inner evolution of $\boldsymbol{x}(t)$ as well as the "Skorohod regulation" while on boundary facets, which will be achieved by the control $\boldsymbol{u}$.

(d) A similar formulation holds for reflected Brownian motion networks; there however, there we use a different regulation constraining mechanism, and a nonseparable quadratic large deviations functional –see below.

(e) The last condition expresses the intuitively clear fact that the asymptotic content process $\boldsymbol{x}(t)$ cannot maintain itself on the boundary (under the catastrophe rates) unless its uncontrolled velocity pushes against it (in the case when S is the positive cone, this is equivalent to $\sum_{s(k)=i} u_k < 1$ if $x_i = 0$).

Definition 6 *An "equilibrium strategy" is a pair* $(\boldsymbol{u}, \bar{\boldsymbol{r}})$ *such that both the controller and nature stand to lose if they deviate from it.*

By extrapolating from the work of Atar, Dupuis and Schwartz on maximum time large deviations control, we arrive at the following "Laplace principle":

Conjecture 2 *The large deviations network games (13) are scaled limit points of risk sensitive stochastic control problems for corresponding renewal regulated networks, with target sets which do not include the attractive center of the fluid field.*

3 The solution of the large deviations network game in the interior of the state space

We construct here the solution of the large deviations network game on a segment of the bicaracteristic contained in the interior of the state space. Differential games are solved by a method due to Isaacs.

One constructs a pre-Hamiltonian H

$$H(\boldsymbol{x}, \boldsymbol{p}, \boldsymbol{u}, \bar{\boldsymbol{r}}) = h(\boldsymbol{x}) - \boldsymbol{u}\, \boldsymbol{L}(\boldsymbol{x}, \bar{\boldsymbol{r}}) + \boldsymbol{p}\; A\; (\bar{\boldsymbol{r}} \bullet \boldsymbol{u})$$

combining the objective and a linear combination of the constraints, given weights p_i called "costates". The costate variables p_i satisfy "adjoint" equations $\frac{dp_i}{dt} = -\frac{\partial}{\partial x_i} H$ and **end** "transversality" condition of proportionality to the gradient of $g(\boldsymbol{x})$, where $g(\boldsymbol{x}) = 0$ denotes the boundary on which the equilibrium path ends. Along the equilibrium path $\boldsymbol{x}, \boldsymbol{p}$ the Hamiltonian is then minimized in $\boldsymbol{u}$ and maximized in $\bar{\boldsymbol{r}}$ (for each fixed values of $\boldsymbol{x}$ and $\boldsymbol{p}$) (due to linearity in $\boldsymbol{u}$ and the convexity in $\bar{\boldsymbol{r}}$, the two operations maybe carried out here in any order), resulting in an Hamiltonian $H^{**}(\boldsymbol{x}, \boldsymbol{p})$ which must be identically 0, along any equilibrium solution $(\boldsymbol{x}(t), \boldsymbol{p}(t))$. This results in the following sytem for the bicharacteristics:

$$\begin{aligned}
\dot{\boldsymbol{x}}(t) &= A\,(\bar{\boldsymbol{r}}(t) \bullet \boldsymbol{u}(t)) \\
\boldsymbol{x}(0) &= \boldsymbol{x} \\
\dot{\boldsymbol{p}}(t) &= -\frac{\partial}{\partial \boldsymbol{x}} H \qquad (14) \\
\boldsymbol{p}(T) &\propto \frac{\partial}{\partial \boldsymbol{x}} g \\
H^{**}(\boldsymbol{x}(t), \boldsymbol{p}(t)) &= \min_{\boldsymbol{u}} \max_{\bar{\boldsymbol{r}}} H(\boldsymbol{x}, \boldsymbol{p}, \boldsymbol{u}, \bar{\boldsymbol{r}}) = 0 \qquad (15)
\end{aligned}$$

Furthemore, in problems with enough smoothness, $\boldsymbol{p} = \frac{\partial}{\partial \boldsymbol{x}} V$ and the last equation above becomes:

$$H^{**}(\boldsymbol{x}, \frac{\partial}{\partial \boldsymbol{x}} V) = 0 \qquad (16)$$

Consider now the problem of sequencing/routing for the multiclass networks previously defined (with activities having at most one starting end and the stations being of three possible types).

We will find it useful to express the pre-Hamiltonian in terms of the **activity indices**

$$\boldsymbol{\sigma} = \boldsymbol{p} A$$

Thus, $\sigma_k = -p_{s(k)} + \sum_{i \in e(k)} p_i = -p_{s(k)} + P_k)$ where $P_k = \sum_{i' \in e(k)} p_{i'}$, generalizing the indices found in [5].

Taking advantage that the Lagrangian is separable and splitting the pre-Hamiltonian into parts due to uncontrolled arrivals, routing, class controlled and sequencing activities, we find:

$$\begin{aligned}
H(\boldsymbol{x}, \boldsymbol{p}, \boldsymbol{u}, \bar{\boldsymbol{r}}) &= h(\boldsymbol{x}) - \sum_{k \in K_a} L_k(\bar{\lambda}_k)) - \sum_{k \in K_c} u_k\, L_k(\bar{\mu}_k) + \boldsymbol{p}\; A\; \tilde{\boldsymbol{r}}(t) \\
&= h(\boldsymbol{x}) + \sum_{k \in K_a} (\sigma_k \bar{\lambda}_k - L_k(\bar{\lambda}_k)) + \sum_{i \in I_r} \big(\bar{\mu}_i(\sum_{s(k)=i} \sigma_k u_k) - L_i(\bar{\mu}_i)\big) \\
&+ \sum_{j \in J_e} \sum_{i \in C(j)} u_i \sum_{s(k)=i} \big(\sigma_k \bar{\mu}_k - L_k(\bar{\mu}_k)\big) + \sum_{k \in K_s} u_k \big(\sigma_k \bar{\mu}_k - L_k(\bar{\mu}_k)\big)
\end{aligned}$$

Letting κ_i, κ_k denote the convex conjugates of L_i, L_k, we find after optimizing $\bar{\boldsymbol{r}}$ that the folowing **convex duality** relations hold between the catastrophe rates and the class indices:

$$\boxed{\bar{r}_k = \kappa_k'(\sigma_k), k \in K_a \cup K_e \cup K_s, \quad \bar{\mu}_i = \kappa_i'(\sum_{s(k)=i} \sigma_k u_k), i \in I_r}$$

and the "semi-pre-Hamiltonian" $H^*(\boldsymbol{x},\boldsymbol{p},\boldsymbol{u}) = \max_{\bar{\boldsymbol{r}}} H^*(\boldsymbol{x},\boldsymbol{p},\boldsymbol{u},\bar{\boldsymbol{r}})$ is:

$$H^*(\boldsymbol{x},\boldsymbol{p},\boldsymbol{u}) = h(\boldsymbol{x}) + \sum_{k\in K_a} \kappa_k(\sigma_k) + \sum_{i\in I_r} \kappa_i(\sum_{s(k)=i} \sigma_k u_k)$$
$$+ \sum_{j\in J_e} \sum_{i\in C(j)} u_i\, R^{(i)} + \sum_{k\in K_s} u_k\, \kappa_k(\sigma_k) \tag{17}$$

where $R^{(i)} = \sum_{s(k)=i} \kappa_k(\sigma_k)$.

Note: In comparison with the uncontrolled structure of the Hamiltonian $H(\boldsymbol{x},\boldsymbol{p}) = h(\boldsymbol{x}) + C(\boldsymbol{p})$ which may be gleaned from [61], we discover here an additional structure, due to the extra types of control we considered.

Let now $k(i)$ be the activity achieving the minimum of $\min_k P_k$ over each class. Minimization in $\boldsymbol{u}$ (using the monotonicity of the cumulant generating functions) reveals:

Proposition 1 *Under assumptions A-C, the saddle solution of a large deviations network game must be such that:*

1. *A sequencing server $j \in J_s$ will only serve activities $k(j)$ achieving a minimal negative priority index:*

$$\min_{S(s(k))=j} \boxed{\kappa_k(\sigma_k)} \tag{18}$$

within the station j, and idle if none exists (this generalizes the priority indices found in [5], [8]).

2. *A class-controlled server $j \in J_e$ will only serve classes $i(j)$ achieving a minimal negative priority index:*

$$\min_{S(i)=j} \boxed{R^{(i)} = \sum_{s(k)=i} \kappa_k(\sigma_k)} \tag{19}$$

within the station j, and idle if none exists.

3. *A "routing" server $j \in J_r$ may only rout to the activity $k(i)$ of classes i that achieve a minimal negative priority index:*

$$\min_{S(i)=j} \boxed{\kappa_i(\sigma_k(i))} \tag{20}$$

within the station j, and idle if none exists.

Notes:

1. The equation (15) of constancy of the optimized Hamiltonian applied with the optimal $\boldsymbol{p}$ yields a "**conservation**" law:

$$H^{**}(x,p) \;=\; h(\boldsymbol{x}) + \sum_{k\in K_a} \kappa_k(\sigma_k) + \sum_{j\in J_s} \kappa_{k(j)}(\sigma_{k(j)}) + \sum_{j\in J_r} \kappa_{i(j)}(\sigma_{k(i(j))}) + \sum_{j\in J_e} R^{(i(j))}$$

where $k(j)/i(j)$ denote the activity/class chosen in each station. In the case of exponential transitions, when $\kappa_k = \bar{r}_k - r_k$, this becomes further a law of the type:

$$H^{**}(x,p) \;=\; h(\boldsymbol{x}) + \sum_{k \in K_a} (\bar{\lambda}_k - \lambda_k) + \sum_{j \in J} (\bar{\mu}_{k(j)} - \mu_{k(j)}) = 0$$

In the case of 0 holding costs, this represents a **global conservation law** between the original sum of the original active rates and the sum of the active "catastrophe" rates, which generalizes the well known exchange of the arrival/departure rates during the filling up of the M/M/1 queue. For a multidimensional generalization, and **local conservation laws** see the "shorting equations" in [10] and below.

2. The complete solution of a large deviations network game may require considerable further work, without being necessarily more complicated conceptually. First, one needs to extend the proposition above to the case of bicharacteristic segments staying within one boundary facet. It turns out that this only entails additional upperbounds on the total efforts per class u_i, which change only slightly the Proposition 1 above. Then, transversality laws need to be developped for moving from one boundary facet to another -see for example [7], [9]. Finally, depending on the function g which defines the boundary, there will be additional final transversality conditions, which will depend on where the bicharacteristic trajectory ends. Furthemore, there may be intervening "switching surfaces", involving boundary conditions which depend on the nature of the surfaces. Instead of a formal treatment, we refer to some illustrative examples in sections 4, involving a switching surface for which the end conditions on one side provide the start conditions on the next, and **??**, which considers the transversality laws needed for moving from one boundary facet to another with codimension 1.

4 Optimal sequencing of a tandem with linear holding costs

We consider now the problem of scheduling networks with infinite buffers (possibly unstable, when uncontrolled) so that integrated linear holding costs $h(\boldsymbol{x}) = \sum_i c_i x_i - c$ plus final costs $G(\boldsymbol{x}) =$ are minimized and with catastrophe described as the first time when one of the buffers becomes 0. It turns out here that the time evolution of the activity indices σ coincides with that found in fluid control; however, the feedback picture connecting $\boldsymbol{u}$ to $\boldsymbol{x}$ will be considerably more complicated.

Example: Consider a (possibly unstable) tandem with two (servicing) stations, three Erlangian activities, and holding cost $h(\boldsymbol{x}) = c_1 x_1 + c_2 x_2 - c$, with catastrophe described as the first time when one of the buffers becomes 0. The control problem is:

$$\begin{array}{rcl}
\min_{\boldsymbol{u}} \max_{\boldsymbol{r}} \; G(\boldsymbol{x}_T) & + & \int_0^T \left(\sum c_i\, x_i(t) - c - n_0 \lambda l(\frac{\bar\lambda}{\lambda}) - u n_1 \mu_1 l(\frac{\bar\mu_1}{\mu_1}) - n_2 \mu_2 l(\frac{\bar\mu_2}{\mu_2})\right) dt \\
\dot x_1(t) & = & \bar\lambda - u(t)\bar\mu_1 \\
\dot x_2(t) & = & u(t)\bar\mu_1 - \bar\mu_2 \\
0 & \le & u(t) \le 1 \\
\boldsymbol{x}(t) & \ge & 0, \qquad \boldsymbol{x}(0) = \boldsymbol{x}, \qquad \mathbf{min}_i\, x_i(T) = \mathbf{0}
\end{array}$$

We construct the Hamiltonian H combining the objective and a linear combination the constraints:

$$H = h(\boldsymbol{x}) - n_0 \lambda l(\frac{\bar\lambda}{\lambda}) - u n_1 \mu_1 l(\frac{\bar\mu_1}{\mu_1}) - n_2 \mu_2 l(\frac{\bar\mu_2}{\mu_2}) + p_1(\bar\lambda - u\bar\mu_1) + p_2(u\bar\mu_1 - \bar\mu_2)$$

Maximization of the Hamiltonian in $\bar{\boldsymbol{r}}$ yields the formulas $\bar\lambda = \lambda e^{p_1/n_0}$, $\bar\mu_1 = \mu_1 e^{(p_2-p_1)/n_1}$, $\bar\mu_2 = \mu_2 e^{-p_2/n_2}$ After the maximization, the Hamiltonian becomes

$$H^*(\boldsymbol{x}, \boldsymbol{p}, \boldsymbol{u}) = h + n_0(\bar\lambda - \lambda) + u n_i(\bar\mu_1 - \mu_1) + n_2(\bar\mu_2 - \mu_2)$$

The linear cost structure implies that

$$p_i(t) = c_i(T - t) + \delta_i$$

The decision is whether to serve or to idle in the first station, and the index equation shows that idling must stop when

$$c_1(\sigma_1) = n_1 \mu_1 (e^{(p_2-p_1)/n_1} - 1) \le 0, \qquad \text{i.e. when } p_2 \le p_1$$

In terms of p_i, these equations are therefore identical with those of [5] (under a different objective!). However, the exponential instead of linear dynamics will lead to a nonlinear switching curve.

Following the fluid solution, we start to investigate first the case of paths ending on $x_2 = 0$, under the assumption that $u = 1$ is optimal. Thus, the dynamics is:

$$x_1' = \lambda e^{\frac{p_1}{n_0}} - \mu_1 e^{\frac{p_2 - p_1}{n_1}}, \qquad x_2' = \mu_1 e^{\frac{p_2-p_1}{n_1}} - \mu_2 e^{-\frac{p_2}{n_2}} \tag{21}$$

Integrating (21), we find that:

$$\begin{array}{rcl}
x_1(t) - x_1 & = & \frac{\lambda n_0}{c_1}(e^{\frac{p_1(0)}{n_0}} - e^{\frac{p_1(t)}{n_0}}) - \frac{\mu_1 n_1}{c_2 - c_1}(e^{\frac{p_2(0)-p_1(0)}{n_1}} - e^{\frac{p_2(t)-p_1(t)}{n_1}}) \\
 & = & \frac{n_0}{c_1}(\bar\lambda(0) - \bar\lambda(t)) - \frac{n_1}{c_2 - c_1}(\bar\mu_1(0) - \bar\mu_1(t)) \\
x_2(t) - x_2 & = & \frac{\mu_1 n_1}{c_2 - c_1}(e^{\frac{p_2(0)-p_1(0)}{n_1}} - e^{\frac{p_2(t)-p_1(t)}{n_1}}) - \frac{\mu_2 n_2}{c_2}(e^{-p_2(t)} - e^{-p_2(0)}) \\
 & = & \frac{n_1}{c_2 - c_1}(\bar\mu_1(0) - \bar\mu_1(t)) - \frac{n_2}{c_2}(\bar\mu_2(t) - \bar\mu_2(0))
\end{array}$$

Looking now for the switching curve, we will assume that at time $t=0$ we are on it, so that $p_2(0)-p_1(0)=0$ (and $\bar{\mu}_1(0)=\mu_1$). At the final time T, taking also into account the Hamiltonian conservation $c_1x_1(T)-c=n_0(\lambda-\bar{\lambda}(T))+n_1(\mu_1-\bar{\mu}_1(T))+n_2(\mu_2-\bar{\mu}_2(T))$ we obtain then

$$\begin{aligned}
c_2x_2 &= -n_1\frac{c_2}{c_2-c_1}(\bar{\mu}_1(0)-\bar{\mu}_1(T))+n_2(\bar{\mu}_2(T)-\bar{\mu}_2(0)) \\
&= -\frac{\mu_1n_1c_2}{c_2-c_1}(1-e^{\frac{\delta_2-\delta_1}{n_1}})+\mu_2n_2(e^{\frac{-\delta_2}{n_2}}-e^{\frac{-c_2T-\delta_2}{n_2}}) \\
c_1x_1-c &= n_0(\lambda-\bar{\lambda}(0))+n_2(\mu_2-\bar{\mu}_2(T))+n_1\frac{c_2}{c_2-c_1}(\bar{\mu}_1(0)-\bar{\mu}_1(T)) \\
&= \lambda n_0(1-e^{\frac{c_1T+\delta_1}{n_0}})+\mu_2n_2(1-e^{\frac{-\delta_2}{n_2}})+\frac{\mu_1n_1c_2}{(c_2-c_1)}(1-e^{\frac{\delta_2-\delta_1}{n_1}})
\end{aligned}$$

Note: An interesting particular case is when $G(\boldsymbol{x})$ coincides with the remaining continuation payoff until $(0,0)$ along the boundary. Then, using $p_1(T)=\frac{\partial G(x_1(T))}{\partial x_1}$ and $H(\boldsymbol{x}(T))=0$ and letting T_i denote the emptying time of class i, we find that $\delta_2=0, \delta_1=c_1(T_1-T_2)-\frac{c}{\mu_1-\lambda}$ (when $c=0$ we recover the formula $p_i(t)=c_i(T_i-t)$ of [5]). The two equations become then:

$$\begin{aligned}
x_2 &= \frac{\mu_1n_1}{c_2-c_1}(e^{\frac{(c_1-c_2)T}{n_1}}-1)+\frac{\mu_2n_2}{c_2}(1-e^{\frac{-c_2T}{n_2}}) \\
x_1 &= \frac{c}{c_1}+\frac{\lambda n_0}{c_1}(1-e^{\frac{c_2T}{n_0}})+\frac{\mu_1n_1c_2}{c_1(c_2-c_1)}(1-e^{\frac{(c_1-c_2)(T)}{n_1}})
\end{aligned}$$

Parametrically, in terms of $a=e^{c_2T}$, this becomes:

$$x_2=\frac{\mu_1n_1}{c_2-c_1}(a^{(\frac{c_1}{c_2}-1)/n_1}-1)+\frac{\mu_2n_2}{c_2}(1-a^{-1/n_2})$$
$$x_1=\frac{c}{c_1}+\frac{\lambda n_0}{c_1}(1-a^{1/n_0})+\frac{\mu_1n_1c_2}{c_1(c_2-c_1)}(1-a^{(\frac{c_1}{c_2}-1)/n_1})$$

In the limit $n_i\to\infty$ we get (using $a^x-1\approx T\,c_2\,x$)

$$x_2\approx(-\mu_1+\mu_2)T \qquad x_1\approx\frac{c}{c_1}+(-\lambda\frac{c_2}{c_1}+\mu_1\frac{c_2}{c_1})T$$

and we rederive thus the fluid result of [5] obtained when $c=0$.

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